Random allocation of bundles*

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Abstract

For the case in which agents have arbitrary preferences over bundles of indivisible objects and monetary transfers are impermissible, I propose a randomized mechanism that generalizes the well-known probabilistic serial mechanism. This generalized mechanism returns an ordinally efficient assignment. It is asymptotically strategy-proof, asymptotically envy-free, and satisfies equal treatment of equals. I also propose a cardinal mechanism, which returns a randomized assignment that approximates a competitive equilibrium with equal income. It satisfies ex-ante envy-freeness and efficiency, and asymptotically envy-freeness. Moreover, this mechanism is strategy proof in the continuum.

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1 Introduction

At least since the publication of Shapley and Scarf (1974), economists have been interested in assigning indivisible objects to agents without the use of monetary transfers. This problem has received renewed attention because of its applications to such problems as school choice and housing assignments (Abdulkadiroglu and Sönmez (1998) and Chen and Sönmez (2002)). Among the many mechanisms considered, one stands out: the probabilistic serial mechanism (PS) (Bogomolnaia and Moulin (2001)). This mechanism has a simple, intuitive description. Agents nibble away at objects to determine probability shares over the objects that agents are awarded. Moreover, as Bogomolnaia and Moulin (2001) show, it possesses the following attractive properties:

1. PS produces ordinally efficient assignments.
2. PS is weakly strategy-proof.
3. PS is envy-free and therefore satisfies equal treatment of equals.

An alternative to the PS is the random priority mechanism (RP), which is widely used in practice and, unlike PS, is strategy-proof. RP can be generalized to environments in which agents consume bundles, while preserving strategy-proofness and weak envy-freeness. Unlike PS, RP is not ordinally efficient (Manea (2009)).

The PS mechanism is limited to cases in which each agent is interested in obtaining at most one object. This prevents its use in environments that require allocating bundles of objects to agents.[1] The main contribution of this paper is to generalize the PS to such settings and to show that it is asymptotically equivalent to RP. The result generalizes the result in Che and Kojima (2010). I call this generalized mechanism the *bundled probabilistic serial mechanism* (BPS). It has the following features:

- It produces ordinally efficient assignments.
- It is asymptotically strategy-proof.
- It is asymptotically envy-free.[2]
- It satisfies equal treatment of equals.
- The unit-demand case reduces to the standard PS mechanism.

Kojima (2009) and Budish et al. (2013) have used probabilistic assignments to allocate indivisible objects to agents when monetary transfers are not allowed. All of the authors cited thus far limit attention to allocations that are representable as probabilities over single objects. In the following paragraphs, I argue that this restriction should be relaxed.

In the one sided matching literature, an expected assignment determines the marginal probability that an agent receives an object. To cover the case of agents with multi-unit demand, Kojima (2009) and Budish et al. (2013) allow for the sum of probabilities that an agent receives single objects to be greater than one. To be able to use any expected assignment in practice, one must be able to represent the expected assignment as a lottery over deterministic assignments in which agents do not overconsume objects. Such an expected assignment is called *implementable*. One major issue with defining probability shares over single objects is that agents may prefer one implementation of an expected

[1] Some examples of such environments are assigning courses to students, medical-student couples to hospital residencies, and siblings to schools and dormitories. In these examples, a group of agents, namely a couple and or a set of siblings, could be viewed as a single agent interested in obtaining multiple objects.

[2] This should be contrasted with an impossibility theorem in Kojima (2009). That theorem rules out the existence of a mechanism that is weakly strategy-proof, envy-free, and ordinally efficient.

assignment over others. This is because different implementations result in different lotteries over bundles. The following example shows this point.

Example 1: *There are two agents, 1 and 2, and four objects, $\{a,b,c,d\}$. The agents' preferences are as follows:*
$\{a,b\} \succ_1 \{c,d\} \succ_1 \{a,c\} \succ_1 \{b,d\} \succ_1 \{a\} \succ_1 \{c\} \succ_1 \{b\} \succ_1 \{d\} \succ_1 \emptyset \succ_1$ *all other bundles.*
$\{c,d\} \succ_2 \{a,b\} \succ_2 \{b,d\} \succ_2 \{a,c\} \succ_2 \{b\} \succ_2 \{d\} \succ_2 \{a\} \succ_2 \{c\} \succ_2 \emptyset \succ_2$ *all other bundles.*

Consider an expected assignment in which agents consume single objects a,b,c, and d all with probability one-half.[3] *Consider an implementation of this assignment in which, with probability one-half, agents 1 and 2 are allocated $\{a,c\}$ and $\{b,d\}$, respectively; and with probability one-half they are allocated $\{b,d\}$ and $\{a,c\}$, respectively. This implementation is Pareto-dominated by the following implementation: with probability one-half agents 1 and 2 are allocated $\{a,b\}$ and $\{c,d\}$, respectively; and with probability one-half they are allocated$\{c,d\}$ and $\{a,b\}$, respectively.*

In cardinal settings in which the agents' utility functions are linear, this restriction does no harm. This is because all implementations of the same expected assignment result in the same expected utility for agents. In the course-scheduling application, this assumption is not plausible,[4] because the presence of complementarities would violate it.[5] For these reasons, in order to take into account agents with arbitrary preferences over bundles, I study expected assignments when probabilities are defined over bundles.

When probabilities are defined over single objects, implementability is trivial. In this case, an expected assignment is implementable if no object is overconsumed (Kojima and Manea (2010)). However, when probabilities are defined over bundles, implementability is not trivial. One of the contributions of this paper is to characterize the set of implementable expected assignments when probabilities are defined over bundles.

Kojima (2009) offers a generalization of the PS when agents have linear cardinal preferences. In Kojima (2009)'s generalization of the PS mechanism to the case of multi-unit demand agents, probability shares from multiple objects that are available are allocated to agents in the same period of time and at the same speed. This mechanism may not result in an ordinally efficient assignment when preferences are nonlinear. Moreover, with general preferences, it is neither envy-free nor strategy-proof. Example 5 in section 3 shows that the algorithm might result in Pareto-inefficient assignments. Budish et al. (2013) offer a similar generalization of the PS mechanism when the designer has some restrictions on the assignment of objects to agents.

To the best of my knowledge, Budish (2011) is the only paper that handles agents with arbitrary preferences over bundles when monetary transfer is not allowed. Inspired by Varian (1974), Budish (2011) proposes a deterministic mechanism, called A-CEEI, for allocating bundles of indivisible objects to agents. A-CEEI chooses an allocation that is the competitive equilibrium of an approximate economy in which agents have roughly equal amounts of money. A-CEEI is strategy-proof in the continuum economy. However, the allocation may not clear the economy considered.[6] The allocation is not pareto-efficient except in an approximate sense. Furthermore, it is envy-free in an approximate sense only. Finally, A-CEEI might violate equal treatment of equals; for example, this is because two agents with identical preferences might be given different budgets.

The tool that has enabled me to design a mechanism (BPS) which performs better in some sense compared to A-CEEI is randomization. I use randomization to design a probabilistic cardinal mechanism, P-CEEI, which asymptotically returns an allocation

[3] *The sum of probabilities is more than one so agents consume more than one object in some realizations of the lottery.*

[4] Students' marginal utility for an additional course could depend on courses they already have in their basket. For example, whether the additional course overlaps with courses already in the basket would matter.

[5] In course scheduling, complementarities in preferences are present. The data (see Kojima et al.(2010)) suggest that in the problem of assigning medical-student couples to hospitals, there are complementarities in couples' preferences over pairs of hospitals.

[6] Therefore, it is impossible to use A-CEEI in environments with no flexibility in the supply.

that is the competitive equilibrium of the economy in which agents have equal budgets. This mechanism inherits asymptotic ex-post fairness, efficiency, and strategy-proofness from the CEEI mechanism.[7]

The outline of the paper is as follows. In section 2, I describe notation and the ordinal model. A description of my generalization of the PS mechanism namelyy (BPS), along with some other mechanisms is provided in section 3. In section 4, I present the properties of the BPS mechanism. In section 5, I describe the cardinal model and present a cardinal mechanism. Concluding remarks are in section 6.

2 Ordinal set up

The primitives of an **ordinal economy** are $(N, G, (n_a)_{a\in G}, (\prec_i)_{i\in N})$ where N is a finite set of **agents**, G is a finite set of **objects**, n_a is the number of copies of object $a \in G$, and $\prec_i$ is agent i's **preference ranking** over bundles of objects. A **bundle** is a vector in $(\mathbb{N} \cup \{0\})^{|G|}$ which specifies the number of copies of each object.[8] For each bundle S and object $a \in G$, let $n_a(S) \in \mathbb{N} \cup \{0\}$ be the number of copies of a in S. It must be the case that $n_a(S) \leq n_a$. Let $\emptyset = \{0\}^{|G|}$ be the bundle with no objects. Let $\mathbf{B}$ be the set of bundles in this economy. I assume preferences are strict. I do not assume that agent neccessarily prefer a bigger bundle.[9] Assume $S \succ_i T$ if agent i prefers bundle S over bundle T. If $\emptyset \succ_i S$, then agent i does not find bundle S acceptable. Let $S \succeq_i T$ if $S = T$ or $S \succ_i T$. For each agent i, let P_i be the set of all possible ordinal preferences of agent i and denote the set of preference profiles by $P = (P_i)_{i\in N}$. A typical element of P is presented by $\succ$.

An **expected assignment** determines the marginal probability that an agent receives a bundle. In this paper, I define expected assignments as a function from the set of agent bundle pairs to the interval $[0, 1]$. Let $N \times \mathbf{B}$ be the set of agent-bundle pairs. Formally, an expected assignment is a function $x : N \times \mathbf{B} \to [0, 1]$. Given $(i, B) \in N \times \mathbf{B}$, then $x(i, B)$ is the marginal probability that agent i receives bundle B. An expected assignment x is **deterministic** if the range of x is $\{0,1\}$.

A deterministic assignment is **implementable** if each agent is allocated at most one bundle and no object is overconsumed. In such assignments, some agents may allocated no non-empty bundle.

Definition 2.1. *A deterministic assignment x is implementable if it satisfies the following two conditions:*

- *For all agents $i \in N$ and for all distinct pairs of bundles $B, B' \in \mathbf{B}$, it must be the case that $x(i, B)x(i, B') = 0$.*
- *For all objects $a \in G$, $\sum_{i\in N} \sum_{B\in\mathbf{B}} n_a(B)x(i, B) \leq n_a$.*

Let $\mathbf{\Phi}$ be the set of all deterministic implementable assignments in this economy. An expected assignment x is implementable if it can be represented as a probability distribution over deterministic implementable assignments.

Definition 2.2. *An expected assignment x is implementable if, for some implementable deterministic assignments $(x^j)_{j=1}^k$ and some nonnegative real numbers $(\alpha_j)_{j=1}^k$ that satisfy $\sum_{j=1}^k \alpha_j \leq 1$, the following holds:*

$$x = \sum_{j=1}^{k} \alpha_j x^j,$$

[7] See Varian (1947) for a description of CEEI.

[8] In all examples in this paper, when there are no copies of objects, I use sets of objects to represent bundles of objects.

[9] In course scheduling application, a larger bundle of courses may create schedule conflict.

where α_j is the probability of assigning x^j to agents.

The following example shows how to implement an expected assignment.

Example 2: *There are three agents $N = \{1, 2, 3\}$ and three objects $G = \{a, b, c\}$, each with one copy. Consider the following expected assignment:*

- *$x(1, \{b, c\})= x(1, \{b\}) = \frac{1}{3}$, and $x(1, B) = 0$ for all other bundles B.*
- *$x(2, \{a, c\}) = x(2, \{c\}) = \frac{1}{3}$, and $x(2, B) = 0$ for all other bundles B.*
- *$x(3, \{a, b\}) = x(3, \{a\}) = \frac{1}{3}$, and $x(3, B) = 0$ for all other bundles B.*

Assignment x is implementable by a uniform distribution over three deterministic implementable assignments: x^1, x^2, and x^3. In x^1, agent 1 is allocated $\{b, c\}$, agent 3 is allocated $\{a\}$, and nothing is allocated to agent 2. In x^2, agent 2 is allocated $\{a, c\}$, agent 1 is allocated $\{b\}$, and nothing is allocated to agent 3. In x^3, agent 3 is allocated $\{a, b\}$, agent 2 is allocated $\{c\}$, and nothingis allocated to agent 1.

The following proposition characterizes the set of implementable expected assignments. Before presenting the proposition, I define some important notation. Let $N \times \mathbf{B} = \{\omega_1, \omega_2, ..., \omega_p\}$ where each ω_r is an agent-bundle pair. Let $\Phi = \{\phi_1, \phi_2, \phi_3, ..., \phi_f\}$, where each ϕ_s is an implementable deterministic assignment. Implementable expected assignments are convex combinations of elements of Φ; denote the set of these convex combinations by $\Delta(\phi)$. Each function $h : N \times B \to \mathbb{R}$ can be represented as a vector $\overrightarrow{h} \in \mathbb{R}^p$, where

$$\overrightarrow{h} \quad = \quad (h(\omega_1), h(\omega_2), h(\omega_3), ..., h(\omega_p)).$$

This vector is called the **corresponding vector** of h. Let $U = [u(r, s)]_{p\times f}$ be a matrix with the property such that $u(r, s) = 1$ iff $\phi_s(\omega_r) = 1$, and equals zero otherwise. In other words, each row of U corresponds to an agent bundle pair, and each column corresponds to an implementable deterministic assignment; the array is 1 if the agent is allocated the bundle in the assignment; otherwise, it is zero. Let $\overrightarrow{1} \in \mathbb{R}^f$ be a row vector of all ones, that is, $\overrightarrow{1} = \underbrace{(1, 1, 1, ..., 1)}_{f \text{ times}}$. Given two vectors $v = (v_i)_{i=1}^n$ and $v' = (v'_i)_{i=1}^n$, I define the relation$v \leq v'$ to be $v_i \leq v'_i$ for all $1 \leq i \leq n$.

Proposition 2.3. *An expected assignment x is implementable iff for all $\lambda : N \times B \to \mathbb{R}^+ \cup \{0\}$ satisfying $\overrightarrow{\lambda} U \leq \overrightarrow{1}$, then $\overrightarrow{\lambda}.\overrightarrow{x} \leq 1$.*

Proof. See appendix A for the complete proof. One can view λ as the agents' cardinal preferences. Given $(i, B) \in N \times \mathbf{B}$, let $\lambda(i, B)$ be the utility that agent i gets from consuming the bundle $\mathbf{B}$. The contrapositive of the proposition is equivalent to the following statement: the expected assignment x is implementable iff for all cardinal utilities such that the sum of agents' expected utilities is more than 1 under assignment x, then the sum of the agents' utilities for some deterministic implementable assignment is also more than 1.

□

The following example is an application of this proposition.

Example 3: *There are three agents $N = \{1, 2, 3\}$ and three objects $G = \{a, b, c\}$, each with one copy. Consider the following expected assignment:*

- *Let $x(1, \{b, c\}) = \frac{1}{2}$, and $x(1, B) = 0$ for all other bundles B.*

- *Let* $x(2,\{a,c\}) = \frac{1}{2}$, *and* $x(2,B) = 0$ *for all other bundles* B.
- *Let* $x(3,\{a,b\}) = \frac{1}{2}$, *and* $x(3,B) = 0$ *for all other bundles* B.

Assignment x *is not implementable. Consider* λ *as follows:*

- *Set* $\lambda(1,\{b,c\}) = 1$ *and* $\lambda(1,B) = 0$ *for all other bundle* B.
- *Set* $\lambda(2,\{a,c\}) = 1$ *and* $\lambda(2,B) = 0$ *for all other bundle* B.
- *Set* $\lambda(3,\{a,b\}) = 1$ *and* $\lambda(3,B) = 0$ *for all other bundle* B.

Think of $\lambda(i,B)$ *as agent* i*'s cardinal utility from bundle* B. *Note that the sum of agents' expected utilities from allocation* x *is* $\frac{3}{2}$, *but for all implementable deterministic assignments the sum of utilities is less than one.*

Any expected assignment when agents have unit demands is implementable iff no object is overconsumed (Kojima and Manea (2010)). This notion of "no overconsumption" can be generalized to the case of agents with a multi-unit demand.

Definition 2.4. *An expected assignment* x *is* ***feasible*** *if both of the following conditions hold:*

- *For each agent* $i \in N$: $\sum_{S\in\mathbf{B}} x(i,S) \leq 1$.
- *For each object* $a \in G$: $\sum_{i\in N}\sum_{S\in\mathbf{B}} n_a(S)x(i,S) \leq n_a$.

If objects are divisible, I can treat probability shares of bundles as fractions of bundles. Feasibility conditions guarantee that the designer can allocate a fraction of the bundles to agents. The expected assignment presented in Example 3 (in section 4) is feasible but not implementable. It is easy to see that the feasibility conditions are necessary for implementability but they are not sufficient. However, I will show in section 4 that in large assignment problems, any feasible expected assignment comes close to being implementable.

An implementable deterministic assignment is Pareto-efficient if there is no other implementable deterministic assignment in which all agents are weakly betteroff and some agents are strictly better off. An implementable expected assignment is **ex-post efficient** if it can be represented as a probability distribution over Pareto-efficient implementable deterministic assignments. An assignment is **ordinally efficient** if there is no other implementable assignment that all agents would weakly prefer and that some agents would strictly prefer under the first-order stochastic dominance criterion **(FOSD)**.

Definition 2.5. *Agent* i *prefers expected assignment* y *to expected assignment* x *under the FOSD criterion if for all* $B \in \mathbf{B}$,

$$\sum_{S\succeq_i B} x(i,S) \quad \leq \quad \sum_{S\succeq_i B} y(i,S).$$

He strictly prefers y *over* x *under the FOSD criterion if the inequality is strict for some* $B \in \mathbf{B}$.

Definition 2.6. *An implementable expected assignment* x *is ordinally efficient if there is no implementable expected assignment* y *such that (i) all agents prefer* y *over* x *under the FOSD criterion and (ii) some agents strictly prefer* y *over* x *under the FOSD criterion.*

Note that ordinal efficiency is stronger than ex-post efficiency. In settings in which agents have unit-demand preferences, Manea (2008) has identified an ordinal welfare theorem, which can be generalized to the present case.

Proposition 2.7. *Let assignment x be implementable. Assignment x is ordinally efficient if and only if for some agents' cardinal preferences, compatible with their ordinal preferences, x maximizes the total agents' utility subject to the implementability constraint.*

Proof. If y dominates x under the FOSD criterion, then the sum of agents' expected utilities under x is smaller compared to that under y, for all cardinal preferences compatible with the ordinal preferences. Therefore, if x maximizes the sum of agents' expected utilities, then it must be ordinally efficient.

To prove the reverse, I define $T : \Delta(\Phi) \to \mathbb{R}^P$ as follows. Given $x \in \Delta(\Phi)$, consider a function $y : N \times \mathbf{B} \to \mathbb{R}$ as $y(i, B) = \sum_{S \succeq_i B} x(i, S)$ for all $(i, B) \in N \times \mathbf{B}$. Set $T(x) = \overrightarrow{y}$, where $\overrightarrow{y}$ is the corresponding vector of y. This function maps the set of implementable expected assignments to a polyhedron. Ordinally efficient assignments are mapped to the northeast boundary points of this polyhedron. Applying the separating hyperplane theorem, I conclude that for any ordinally efficient and implementable assignment x^*, there is a positive vector $\overrightarrow{q} \in \mathbb{R}^P$ with the property that: $\overrightarrow{q}.T(x) \leq \overrightarrow{q}.T(x^*)$ for all $x \in \Delta(\Phi)$. Given $\overrightarrow{q}$, consider $\lambda : N \times \mathbf{B} \to \mathbb{R}$ such that $\overrightarrow{\lambda} = \overrightarrow{q}$. For all $(i, B) \in N \times \mathbf{B}$, the utility of agent i for bundle $\mathbf{B}$ is defined as follows:

$$u_i(B) \quad = \quad \sum_{S \succeq_i B} \lambda(i, S).$$

This utility function satisfies the condition in the theorem. □

An ordinal mechanism f maps reported preferences to implementable assignments. That is, f is a function $f : P \to \Delta(\Phi)$. Given the agents' preference profile $\succ$ agent $i \in N$ and bundle $B \in \mathbf{B}$, then $f(\succ)(i, B)$ is the marginal probability that agent i receives bundle B. A mechanism is **strategy-proof** if each agent prefers, under the FOSD criterion, the random assignment with truth-telling over the random assignment with a misreport.

Definition 2.8. *Mechanism f is strategy-proof if for all agents $i \in N$, $\succ_i$ & $\succ'_i \in P_i, \succ_{-i} \in P_{-i}$, and bundle $B \in \mathbf{B}$, the following holds:*

$$\textstyle\sum_{S \succeq_i B} f(\succ_i, \succ_{-i})(i, S) \geq \sum_{S \succeq_i B} f(\succ'_i, \succ_{-i})(i, S).$$

An ordinal mechanism is **weakly strategy-proof** if no agent can strictly improve his expected assignment, under the FOSD criterion, by misreporting his preferences.

Definition 2.9. *Mechanism f is weakly strategy-proof if for all $i \in N$, bundle $B \in \mathbf{B}$, $\succ_i \in P_i$, and $\succ_{-i} \in P_{-i}$ there is no $\succ'_i$ such that the following holds:*

$$\textstyle\sum_{S \succeq_i B} f(\succ_i, \succ_{-i})(i, S) \leq \sum_{S \succeq_i B} f(\succ'_i, \succ_{-i})(i, S),$$

with strict inequality for some bundle $B \in \mathbf{B}$.

A mechanism is **envy-free** if agents prefer, under the FOSD criterion, their own expected assignment to any other agent's expected assignment.

Definition 2.10. *Mechanism f is envy-free if for all $i, j \in N$,$\prec \in P$, and bundle $B \in \mathbf{B}$, the following holds:*

$$\textstyle\sum_{S \succeq_i B} f(\succ)(i, S) \geq \sum_{S \succeq_i B} f(\succ)(j, S).$$

A mechanism is **weakly envy-free** if no agent strictly prefers, under the FOSD criterion, another agent's expected assignment to his own expected assignment.

Definition 2.11. *Mechanism f is weakly envy-free if, for all pairs of agents $i, j \in N$ and preference profile $\prec \in P$, the following holds:*

$$\forall B \in \mathbf{B} \; \sum_{S \preceq_i B} f(\succ)(i,S) \leq \sum_{S \preceq_i B} f(\succ)(j,S) \Rightarrow \forall B \in \mathbf{B} \; f(\succ)(i,B) = f(\succ)(j,B).$$

A mechanism satisfies **equal treatment of equals** if all agents with the same reported preferences receive the same expected assignments.

Definition 2.12. *Mechanism f satisfies equal treatment of equals if, for all $i, j \in N$ and $\prec_i = \prec_j$, then $f(\succ)(i,B) = f(\succ)(j,B)$ for all $B \in \mathbf{B}$. This means that agents with the same reported preferences are allocated the same expected assignments.*

3 Mechanisms

In this section, I present two ordinal mechanisms. The first mechanism, **random priority** (RP), is widely used in practice. The second mechanism, the **bundled probabilistic serial mechanism** (BPS), is my proposed generalization of the PS mechanism.

3.1 Random-Priority Mechanism

In the RP agents are ordered uniformly at random. Based on the order assigned, agents are asked, one at a time, to pick their best available bundle. This ordinal mechanism is strategy-proof and weakly envy-free. RP may produce ordinally inefficient assignments (Bogomolnaia and Moulin (2001)). Moreover, as discussed in Budish and Cantillon (2012), the mechanism might result in large ex-post envy. In the following example, I run the RP mechanism.

Example 4: *There are three agents $\{1,2,3\}$ and four objects with single copies*

$$\{a,b,c,d\}.$$

Preferences are defined as follows:

- $\{d,b,c\} \succ_1 \{b,c\} \succ_1 \emptyset \succ_1$ all other bundles.
- $\{d\} \succ_2 \{a,b\} \succ_2 \{a\} \succ \emptyset \succ_2$ all other bundles.
- $\{d\} \succ_3 \{a,c\} \succ_3 \{c\} \succ_3 \emptyset \succ_3$ all other bundles.

RP results in the following expected assignment:

- $x(1,\{b,c,d\}) = \frac{1}{3}$, $x(1,\{b,c\}) = \frac{1}{3}$, *and* $x(1,B) = 0$ *for all other bundles* B.
- $x(2,\{a\}) = \frac{1}{2}$, $x(2,\{a,b\}) = \frac{1}{6}$, $x(2,\{d\}) = \frac{1}{3}$, *and* $x(2,B) = 0$ *for all other bundles* B.
- $x(3,\{a,c\}) = \frac{1}{6}$, $x(3,\{d\}) = \frac{1}{3}$, *and* $x(1,B) = 0$ *for all other bundles* B.

3.2 The Bundled Probabilistic Serial Mechanism

Before defining the BPS mechanism, I present an example that shows why the generalization of the PS mechanism in Kojima (2009) may be inefficient.

Example 5: *Consider the setup in Example 1. If agents are interested in obtaining bundles of size at most two, the generalization in Kojima (2009) would result in an assignment in which agent 1 is allocated $\{a,c\}$ and agent 2 is allocated $\{b,d\}$. This is not* *efficient, since it is Pareto-dominated by an assignment in which agent 1 consumes $\{a, b\}$ and agent 2 consumes $\{c, d\}$.*

Loosely speaking, in the *BPS* mechanism agents are allocated probability shares from their best possible bundle with the same speed, subject to the constraint of implementability.

Here is a formal definition of the mechanism. For all $(i, B) \in N \times \mathbf{B}$ let $I_{\{\omega=(i,B)\}}$ be a deterministic assignment in which agent i is allocated bundle B, and nothing is allocated to other agents. If $B = \emptyset$, set $I_{\{\omega=(i,B)\}} = 0$. Given an initial implementable assignment x, a bundle is available for agent i if adding that bundle with small enough probability to agent $i's$ assignment does not violate the implementability of the allocation. In other words, $x + \epsilon I_{\{\omega=(i,B)\}} \in \Delta(\Phi)$, for some $\epsilon > 0$. Let B_i^x be the best available bundle for agent i. If there is no available bundle for agent i, let B_i^x be the empty bundle. The expected assignment $M(x) \in \Delta(\Phi)$ is defined as:

$$M(x) = x + \epsilon_x \sum_{i \in N} I_{\{\omega=(i,B_i^x)\}},$$

where ϵ_x is the largest positive number such that the assignment $x + \epsilon_x \sum_{i \in N} I_{\{\omega=(i,B_i^x)\}}$ is implementable. If $B_i^x \neq 0$ for some agent $i \in N$, then, since $\Delta(\Phi)$ is convex, such an $\epsilon_x > 0$ exists. If $B_i^x = \emptyset$ for all agents i, set $M(x) = x$. Given the initial allocation x, in order to construct to construct $M(x)$, agents simultaneously eat the best bundle that is available to them as long as the expected assignment is implementable.

I now consider the following sequence exptected assignments:

$$0, M_1(0), M_2(0), M_3(0), ...$$

where $M_k(0) = M(M_{k-1}(0))$. Let τ be the first index such that $M_\tau(0) = M_{\tau-1}(0)$, and set BPS= $M_\tau(0)$. Such a τ exists, since (i) there is a finite number of agents and bundles, (ii) in each step at least one agent changes his bundle, and (iii) the agents who changed their bundles in a given step will not be reallocated probability shares from these bundle in future steps.

In the following example, I apply Proposition 2.3 to run this algorithm.

Example 6: *Consider the setup in Example 5. Starting from the assignment of 0, the BPS allocates to agents 1, 2, and 3 probability shares from bundles $\{d, b, c\}$,$\{d\}$, and $\{d\}$, respectively. Allocating a one-third probability share of these bundles results in an implementable assignment, and further assignment of these bundles results in the overconsumption of object d and hence violates implementability. Therefore, $M_1(0)$ is the allocation that agents get one-third probability shares of their best bundles.*

Since allocating more probability shares of any agent's best bundle would violate implementability, best bundles are no longer available to agents. Simultaneously, they all point to their second-best bundles, which are $\{b, c\}$,$\{a, b\}$, and $\{a, c\}$, respectively. I argue that the expected assignment $M_2(0)$ is constructed by allocating these bundles to agents with probability $\frac{2}{9}$. First, I show that the candidate $M_2(0)$ is implementable. It can be implemented by the five implementable deterministic assignments x^1, x^2, x^3, x^4, and x^5 with corresponding probabilities $\frac{1}{3}, \frac{2}{9}, \frac{1}{9}, \frac{1}{9}$, and $\frac{2}{9}$, respectively. In x^1, agent 1 is allocated $\{d, b, c\}$. In x^2, agent 2 is allocated $\{d\}$ and agent 3 is allocated $\{a, c\}$. In x^3, agent 1 is allocated $\{b, c\}$ and agent 2 is allocated $\{d\}$. In x^4, agent 1 is allocated $\{b, c\}$ and agent 3 is allocated $\{d\}$. In x^5, agent 2 is allocated $\{a, b\}$ and agent 3 is allocated $\{d\}$. Second, I show that $\frac{2}{9}$ is the highest probability share of bundles that one can allocated. I use Proposition 2.3. Consider the following λ: $\lambda(1, \{b, c, d\}) = 1$; $\lambda(2, \{d\}) = \lambda(3, \{a, c\}) = \lambda(1, \{b, c\}) = \lambda(2, \{a, b\}) = \lambda(3, \{d\}) = \lambda(3, \{c\}) = \frac{1}{2}$; and $\lambda(i, B) = 0$ for all other (i, B). Note that λ satisfies the conditions in the proposition and $\lambda.M_2(0) = 1$. Therefore, I cannot allocate probability shares of these bundles to agents any more.

Agent 1 now has no other acceptable bundle. Agents 2 and 3 point to their third and last acceptable bundles, $\{a\}$ and $\{c\}$, respectively. Agent 3 cannot be allocated any of $\{c\}$, since the same λ as in the previous case can be used to show that it would violate implementability. Agent 2 cannot be allocated more than a $\frac{1}{9}+\frac{1}{3}$ probability share of object a; otherwise sum of the probability shares of bundles allocated to agent 2 exceeds 1. To show that adding a $\frac{4}{9}$ probability share of object a to agent 2 makes thte expeced assignment$M_3(0)$, it suffices to show that this is implementable; in fact, it can be implemented by thte five implementable deterministic assignments x^1, x^2, x^3, x^4, and x^5 with corresponding probabilities $\frac{1}{3}$, $\frac{2}{9}$, $\frac{1}{9}$, $\frac{1}{9}$ and $\frac{2}{9}$, respectively. In x^1, agent 1 is allocated $\{d, b, c\}$ and agent 2 is allocated $\{a\}$. In x^2, agent 3 is allocated $\{a, c\}$ and agent 2 is allocated $\{d\}$. In x^3, agent 1 is allocated $\{b, c\}$ and agent 2 is allocated $\{d\}$. In x^4, agent 1 is allocated $\{b, c\}$, agent 3 is allocated $\{d\}$, and agent 2 is allocated $\{a\}$. In x^5, agent 2 is allocated $\{a, b\}$ and agent 3 is allocated $\{d\}$. The algorithm stops and the allocation produced by BPS mechanism is as follows:

- $BPS(1, \{b, c, d\}) = \frac{1}{3}$, $BPS(1, \{b, c\}) = \frac{1}{9}$, *and* $BPS(1, B) = 0$ *for all other bundles* B.
- $BPS(2, \{a\}) = \frac{4}{9}$, $BPS(2, \{d\}) = \frac{1}{3}$, $BPS(2, \{a, b\}) = \frac{2}{9}$, *and* $BPS(2, B) = 0$ *for all other bundles* B.
- $BPS(3, \{d\}) = \frac{1}{3}$, $BPS(3, \{a, c\}) = \frac{2}{9}$, *and* $BPS(3, B) = 0$ *for all other bundles* B.

4 Properties of the BPS

4.1 General Properties of BPS

Remark 4.1. *BPS is a generalization of PS; i.e., when agents point to single objects, BPS and PS are the same.*

Proof. Proof follows from the fact that feasibility and implementability are equivalent when the lottery is over single objects. □

Theorem 1. *BPS produces ordinally efficient assignments and satisfies equal treatment of equals.*

Proof. See appendix B for the proof. □

4.2 Properties of BPS in Large Assignment Problems

To discuss the properties of BPS in large economies, I first formally define large economies and the continuum economy. The definition here of the $q-$economy is similar to its definition in Che and Kojima (2010). For each $q \in \mathbb{N}$, the $\mathbf{q-}$**economy** consists of

$$(N_q, G, (n_a^q)_{a\in G}, (\prec_i)_{i\in N_q}).$$

G is the set of objects. Each object $a \in G$ has (n_a^q) copies. I assume that there exists a $\kappa > 0$ such that agents do not find bundles with sizes more than κ acceptable. This assumption is reasonable in at least some applications. For example, in course scheduling a time restriction limits the size of acceptable bundles. Also, in the problem of assigning siblings to schools and medical-student couples to residencies, the size of bundles demanded does not depend on the size of the economy considered. I assume that the number of copies of each object is at least κ in all q-economies; furthermore, $\lim_{q\to\infty} n_a^q = \infty$ for all $a \in G$. Let $\mathbf{B}^*$ be the set of bundles with size at most κ. The set of agents, N_q, is partitioned into k subsets, Π_θ^q for $1 \le \theta \le k$, where all agents with the same type have the same preference ranking over bundles in the $q-$economy. Agents that are in the set Π_θ^q are called *agents with type θ*. An expected assignment in the $q-$economy is **symmetric** if all agents in the same partition receive the same expected assignments. Formally, x is symmetric if for all $1 \leq \theta \leq k$, $B \in \mathbf{B_q}$, and $i, j \in \Pi_\theta^q$, $x(i, B) = x(j, B)$.

The **continuum economy** is also a model of large economies. I define it to be the limit of $q-$economies as q converges to infinity. The primitives of this economy are:

$$(N^*, G, (n_a^*)_{a \in G}, (m_\theta^*)_{1 \leq \theta \leq k}, (\prec_i)_{i \in N^*}).$$

For each object $a \in G$, there is a mass n_a^* of this object. The set of agents, N^*, is an interval of real numbers partitioned into k intervals $(\Pi_\theta^*)_{1 \leq \theta \leq k}$. Each point in N^* corresponds to an agent. For all $1 \leq \theta \leq k$, the length of Π_θ^* is m_θ^*. Agents do not find bundles of size more than κ acceptable. The set of all bundles with sizes no more than κ is denoted as $\mathbf{B}^*$. For each $1 \leq \theta \leq k$, the agents in Π_θ^* have the same preference ranking of bundles in $\mathbf{B}^*$ as do the agents with type θ in the $q-$economies. An expected assignment in the continuum environment is a function $x : N^* \times \mathbf{B}^* \rightarrow [0, 1]$. The expected assignment is deterministic if the range of x is $\{0, 1\}$. A deterministic assignment is implementable if (i) agents are allocated to at most one bundle, and (ii) for all objects $a \in G$ the measure of agents allocated to bundles that include object a (including all copies of a in all bundles) does not exceed n_a^*. That is, $\int_{N^*} \sum_{B \in \mathbf{B}^*} x(i, B) n_a(B) di \leq n_a^*$ for all $a \in G$. An expected assignment in the continuum economy is implementable if it can be represented as a probability distribution over implementable deterministic allocations. Similarly, I can define feasibility for expected assignments: x is feasible if $\int_{N^*} \sum_{B \in \mathbf{B}^*} x(i, B) n_a(B) di \leq n_a^*$ for all $a \in G$ and for all $i \in N^*$, $\sum_{B \in \mathbf{B}^*} x(i, B) \leq 1$. If agents with the same type are allocated the same expected assignment, the assignment is called symmetric**.** I now show that in the continuum economy, feasibility and implementability are equivalent.

Proposition 4.2. *A symmetric expected assignment is implementable in the continuum economy if and only if it is feasible.*

Proof. Let x be a symmetric and feasible expected assignment, and let

$$\mathbf{B}^* = \{S_1, S_2, S_3, ..., S_m\}.$$

For all $1 \leq \theta \leq k$, assume that agents with type θ are located in the following interval:

$$[\sum_{t=0}^{\theta-1} m_t^*, \sum_{t=0}^{\theta-1} m_t^* + m_\theta^*],$$

where $m_0^* = 0$. Let $\pi = (\pi_\theta)_{1 \leq \theta \leq k} \in \prod_{1 \leq \theta \leq k} [\sum_{t=0}^{\theta-1} m_t^*, \sum_{t=0}^{\theta-1} m_t^* + m_\theta^*]$. Move the agent with position t in the interval $[\sum_{t=0}^{\theta-1} m_t^*, \sum_{t=0}^{\theta-1} m_t^* + m_\theta^*]$ to position $t + \pi_\theta (\mod m_\theta^*)$. That is, if $t + \pi_\theta \leq \sum_{t=0}^{\theta-1} m_t^* + m_\theta *$, then the new position is $t + \pi_\theta$; otherwise, it is $t + \pi_\theta - m_\theta^*$. I construct y^π as follows: for all $1 \leq \theta \leq k$, $i \in \Pi_\theta^*$, and $1 \leq j \leq m$, given the new position of agents, assume that $x(i, S_0) = 0$ and allocate S_j to agents in the interval $[\sum_{t=0}^{\theta-1} m_t^* + \sum_{l=0}^{j-1} x(i, S_l), \sum_{t=0}^{\theta-1} m_t^* + \sum_{l=0}^{j} x(i, S_l)]$. Note that y^π is an implementable deterministic allocation in the continuum economy, and that x can be represented as a lottery over y^π, where π is uniformly distributed in $\prod_{1 \leq \theta \leq k} [\sum_{t=0}^{\theta-1} m_t^*, \sum_{t=0}^{\theta-1} m_t^* + m_\theta^*]$.

□

Throughout this paper, I make the following assumption:

Assumption 4.3. *For each object $a \in G$, $m_a^* \neq 0$. Also, the sequence of the $q-$economy converges to the continuum economy, i.e., $lim_{q\to\infty}\frac{n_a^q}{q} = n_a^*$ and $lim_{q\to\infty}\frac{|\Pi_q^\theta|}{q} = m_\theta^* \in \mathbb{R}$, $\forall 1 \leq \theta \leq k$.*

This assumption says that both the number of copies of each object and the number of agents with each type grow at the same rate as q.

The following proposition shows that, in large economies, any feasible expected assignment comes close to being an implementable assignment.

Proposition 4.4. ***Rounding:*** *Let $(x^q)_{q\in\mathbb{N}}$ be a sequence of feasible expected assignments where $x^q : N \times \mathbf{B}^* \to [0,1]$ is an expected assignment in the $q-$economy. There exists a sequence $(y_q)_{q\in\mathbb{N}}$, where y_q is an expected assignment in the $q-$economy such that for all $\epsilon > 0$, there exists $Q > 0$ that satisfies the following:*

$$\forall \omega \in N_q \times \mathbf{B}^* \text{ and } q > Q,\ |x^q(\omega) - y^q(\omega)| < \epsilon.$$

Proof. See appendix C for the proof, which relies neither on the partitioning of N^q into k subsets nore Assumption 4.3.

□

To show that RP and BPS are asymptotically equivalent, I introduce a naive generalization of the PS mechanism, called NPS. Recall that in PS, indivisible objects are regarded as divisible probability shares, and agents simultaneously, at unit speed and for one unit of time, take probability shares from the best available object. The portion that each agent consumes from an object is the probability that the agent is allocated the object. I can generalize this mechanism to the environment studied here. A natural generalization is to allow agents, for one unit of time at the unit speed, to eat the best bundle whose objects were not previously consumed.

Consider the $(N, G, (n_a)_{a\in G}, (\prec_i)_{i\in N})$ economy. For any object $a \in G$ and any subset of goods $A \subseteq G$, let $m^i(A)$ be agent $i's$ most preferred bundle, consisting of objects in A. Let $m_a^i(A)$ be the number of copies of object a in $m^i(A)$, that is, $m_a^i(A) = n_a(m^i(A))$. Set $m_a(A) = \sum_{i\in N} m_a^i(A)$; note that $m_a(A)$ is the total number of copies of object a in all agents' preferred bundles. Following (Bogomolnaia and Moulin (2001)), the generalization is defined formally by the following steps. For step $v = 0$ and all objects $a \in G$, let $G(0) = G$, $t(0) = 0$, $x_a(0) = 0$, and $NPS(0) = 0$. Given step $v-1$, step v is defined as follows:

- $t_a(v) = \sup\{t \in [0,1] | x_a(v-1) + m_a(G(v-1))(t - t(v-1)) \leq n_a\}$.
- $t(v) = \min_{a\in G(v-1)} t_a(v)$.
- $G(v) = G(v-1) - \{a \in G(v-1) | t_a(v) = t(v)\}$.
- $x_a(v) = x_a(v-1) + m_a(G(v-1))(t(v) - t(v-1))$.
- $NPS(v) = NPS(v-1) + \sum_{i\in N}(t(v) - t(v-1)) I_{\{\omega=(i,(G(v-1))\}}$.

The last step is defined as $v = \min\{v | t(v) = 1\}$.

Given the initial allocation of $NPS(v-1)$, $t(v)$ is the time at which step v ends. Denote the set of objects not exhausted at the end of step v by $G(v)$. Given object $a \in G$, $x_a(v)$ is the amount of consumption from object a until the end of step v. If object a is being consumed in step v, then $t_a(v)$ is the time when object a will be totally consumed. Since there is a finite number of objects, this process stops in a finite number of steps. NPS is the last step's allocation. The expected assignment produced is feasible but it may not be implementable, as the following example shows.

Example 8: *There are three objects $\{a, b, c\}$ and three agents $\{1, 2, 3\}$ with the following preferences:*

- $\{b,c\} \succ_1 \{b\} \succ_1 \{c\} \succ_1 \emptyset \succ_1$ *all other bundles.*
- $\{a,c\} \succ_2 \{a\} \succ_2 \{c\} \succ_2 \{a,b\} \succ_2 \emptyset \succ_2$ *all other bundles.*
- $\{a,b\} \succ_3 \{a\} \succ_3 \{a,c\} \succ_3 \emptyset \succ_3$ *all other bundles.*

NPS produces an assignment in which agent 1 consumes bundle {b,c}, agent 2 consumes bundle {a,c}, and agent 3 consumes {a,b}, all with probability half. In this expected assignment, although no object is overconsumed, the expected assignment is not implementable.

Since the result of this mechanism may not be implementable, we cannot use this mechanism in practice. A **pseudomechanism** is a function that maps ordinal preferences to feasible expected assignments. NPS is a pseudomechanism. The properties of envy-freeness and weak strategy-proofness can be generalized naturally to pseudomechanisms.[10] Despite the implementability issue, NPS satisfies the properties of the standard PS mechanism.

Proposition 4.5. *NPS satisfies the envy-free and weak strategy-proof properties.*

Proof. This proof is omitted since it is an adaptation of Theorem 1 and Proposition 1 in Bogomolnaia and Moulin (2002). □

Since feasibility and implementability coincide for symmetric expected assignments in the continuum economy, BPS and NPS coincide. BPS and NPS can be constructed for the continuum economy using the following steps. For step $v = 0$, let $G^*(0) = G$, $t^*(0) = x^*_a(0) = 0$ for all objects $a \in G$. For each subset of objects $A \subseteq G$, let $m_a(A) = \int_{i \in N^*} m^i_a(A) di$. Given $G^*(v{-}1), t^*(v{-}1),) \; x^*_a(v{-}1)$ for all $a \in G$:

- $t^*_a(v) = sup\{t \in [0,1] | x^*_a(v{-}1) + m_a(G(v{-}1))(t{-}t^*(v{-}1)) \leq m^*_a\}$.
- $t^*(v) = \min_{a \in G(v-1)} t^*_a(v)$.
- $G^*(v) = G^*(v{-}1){-}\{a \in G^*(v{-}1) | t^*_a(v) = t^*(v)\}$.
- $x^*_a(v) = x^*_a(v-1) + m_a(G^*(v{-}1))(t^*(v){-}t^*(v{-}1))$.

The last step is defined as $v = \min\{v | t^*(v) = 1\}$.

Interpretations of $t^*_a(v), t^*(v), G^*(v),$ and $x^*_a(v)$ are the same as those for$t_a(v), t(v), G(v)$, and $x_a(v)$, respectively.

Since NPS^* is symmetric and feasible in all of its steps, it is implementable in all steps. Because of this, the BPS mechanism in the continuum economy, denotaed as BPS^*, is the same as NPS^*.

4.2.1 Asymptotic Equivalence of RP and BPS

Che and Kojima (2010) show that PS and RP are asymptotically equivalent for unit-demand agents. Their result is generalized in Liu and Pycia (2012) asymptotic equivalence result for any two mechanisms that satisfy a number of properties. In this section, I generalize the result of Che and Kojima (2010) to the case of agents with a multi-unit demand. For any mechanism f, let f^q and f^* be the adaptation of f to the $q-$economy and to the continuum economy, respectively.

The following definition formally defines the *convergence* of mechanisms.

[10] Given a pseudo mechanism f, f is envy free if for all $i, j \in N$, i prefers, under the FOSD criterion, his own assignment to agent j's assignment. It is weakly strategy-proof if no agent can strictly improve his assignment, under the FOSD criterion, by misreporting his preferences.

Definition 4.6. *Mechanism f **converges** to f^* if, for all $\epsilon > 0$, there is a $Q > 0$ such that for all $q \geq Q$, $1 \leq \theta \leq k$, $i \in N_\theta^q$, $i^* \in N_\theta*$, and all bundles $B \in \mathbf{B}^*$ $|f^q(i,B) - f^*(i^*,B)| < \epsilon$. Two mechanisms f and g are **asymptotically equivalent** if, for all $\epsilon > 0$, there is a $Q > 0$ such that for all $q \geq Q$, $\forall 1 \leq \theta \leq k$, all $i, j \in N_\theta^q$, and all bundles $B \in \mathbf{B}^*$,*

$$|f^q(i,B) - g^q(j,B)| \;\; < \;\; \epsilon.$$

.

Proposition 4.7. *NPS^q converges to NPS^*.*

Proof. The proof is omitted, since it is also a variation of the proof of Theorem 1 in Che and Kojima (2010).

□

Proposition 4.8. *RP^q converges to NPS^*.*

Proof. The proof is omitted, since it is also a variation of the proof of Theorem 2 in Che and Kojima (2010).

□

Theorem 2. *BPS and RP are asymptotically equivalent.*

Proof. For the complete proof, see appendix D. Propositions 4.7 and 4.8, together with the asymptotic equivalence of BPS and NPS, prove the theorem. I prove that BPS and NPS are equivalent by induction on the steps of the BPS mechanism.

□

Asymptotically strategy-proofness and **asymptotically envy-freeness** are defined in a way that is similar to the definition of strategy-proofness and envy-freeness.

Definition 4.9. *A mechanism f is asymptotically strategy-proof if, for all $\epsilon > 0$, there exists a $Q > 0$ such that, for all $q > Q$, all $i \in N_q$, $\succ_i$ & $\succ_i' \in P_i$, $\succ_{-i} \in P_{-i}$, and $B \in \mathbf{B}^*$, the following holds:*

$$\textstyle\sum_{S \preceq_i B} f_q(\succ_i, \succ_{-i})(i,S) + \epsilon \geq \sum_{S \preceq_i B} f_q(\succ_i', \succ_{-i})(i,S).$$

Mechanism f is asymptotically envy-free if, for all $\epsilon > 0$, there exists a $Q > 0$ such that for all $q > Q$, all pair of agents $i, j \in N_q$, $\succ \in P$, and bundle $B \in \mathbf{B}^*$, the following holds:

$$\textstyle\sum_{S \preceq_i B} f_q(\succ)(i,S) + \epsilon \geq \sum_{S \preceq_i B} f_q(\succ)(j,S).$$

Though BPS is not in general strategy-proof and envy-free, it possesses these properties asymptotically. The following statement, which is a corollary of Theorems 1 and 2, summarizes all properties of the BPS mechanism.

Corollary 4.10. *BPS has the following properties:*

1. *It produces ordinally efficient probabilistic allocations.*
2. *It satisfies equal treatment of equals.*
3. *It is asymptotically strategy-proof.*
4. *It is asymptotically envy-free.*

Proof. Theorem 1 states the first two properties. Theorem 2, along with the fact that RP is strategy-proof, implies the third and fourth.

□

5 A Cardinal Mechanism: P-CEEI

In this section I relax the assumption that preferences are strict.

Varian (1974) proposes a way to allocate indivisible objects when agents have unit demand. His mechanism is called *competitive equilibrium from equal income* (CEEI). In CEEI the designer endows agents with equal amounts of a fake budget and finds the competitive equilibrium.

The difficulty in generalizing this idea to the case of agents with a multi unit demand is that in general prices that clear the economy being considered may not exist. Budish (2011) has generalized this idea for the case of agents with multi-unit demand. His ordinal mechanism (A-CEEI) requires increasing the number of objects, and it may allocate different fake budgets to different agents. I generalize the Varian (1974) mechanism without increasing the number of objects or allocating discriminatory budgets. The idea is that when I consider random assignments of bundles of objects, market-clearing prices exist. This random assignment is feasible, but unfortunately it may not be implementable. I therefore use the rounding proposition to make this assignment implementable.

Example 7: *Consider two agents,* 1 *and* 2*, and four objects* $\{a, b, c, d\}$*, each with one copy. Agents' preferences over bundles of objects are as follows:*

- $u_1(\{a, b\}) = u_1(\{c, d\}) = 3$*, and* $1 \geq u_1(B) \geq 0$ *for all other bundles* B*.*
- $u_2(\{a, c\}) = u_2(\{b, d\}) = 4$*, and* $1 \geq u_2(B) \geq 0$ *for all other bundles* B*.*

Agents are endowed with one unit of a fake budget. If $p_a = p_b = p_c = p_d = \frac{1}{2}$*, consider the following random assignment. Agent* 1 *is allocated* $\{a, b\}$ *and* $\{c, d\}$*, each with probability one-half; while agent* 2 *is allocated* $\{b, d\}$ *and* $\{c, a\}$*, each with probability one-half. This random assignment is feasible. Note that there is no price vector and corresponding deterministic allocation that would clear the market.*

A-CEEI is approximately efficient, strategy-proof in the continuum economy, and is approximately ex-post fair. My generalization of CEEI, P-CEEI, is strategy-proof in the continuum, approximately efficient, ex-ante envy-free, and ex-post envy-free with a high probability. More importantly, P-CEEI does not require adding additional copies of objects to the economy.

5.1 Setup for the Cardinal Economy

The primitives of the cardinal economy are $(N; G; (n_a)_{a\in G}, (u_i)_{i\in N})$, where N, G, and $(n_a)_{a\in G}$ are defined as in the ordinal case. The utility that agent i receives from bundle B is $u_i(B)$. I assume that $u_i(\emptyset) = 0$ for all agents$i \in N$. If i does not find bundle B acceptable, then assume $u_i(B) = -\infty$. The $q-$economy in the cardinal environment is defined as in the ordinal setting, except for the fact that, agents with similar preferences, are not partitioned into subsets. In the $q-$economy, agents do not find bundles of size more than κ acceptable. I assume that there is a uniform bound on the utility that agents have in the $q-$economy, for all $q \in N$. Otherwise, preferences are arbitrary. Assume that agents rank lotteries over bundles with their expected utilities.

5.2 Description of the mechanism

Consider the economy consisting of $(N, G, (n_a)_{a\in G}, (u_i)_{i\in N})$. I endow each agent with one unit of a fake budget. Given price vector $p = (p_a)_{a\in G}$, an agent's demand is a lottery over bundles that gives the agent the highest possible expected utility among all affordable lotteries.

Definition 5.1. *Given a probabilistic assignment x, let $x(i) : \mathbf{B} \to [0,1]$ be agent i's assignment, defined as $x(i)(B) = x(i,B)$. An agent $i \in N$ demands $x(i)$ if it is the solution to the following problem:*

$$\max_{y:\mathbf{B}\to[0,1]} \sum_{B\in\mathbf{B}} y(B)u_i(B) \text{ subject to}$$
$$\sum_{B\in\mathbf{B}} y(B) \sum_{a\in G} n_a(B)p_a \leq 1.$$

Proposition 5.2. *There exists a nonnegative price vector p and a feasible expected assignment x such that agent i demands $x(i)$. Moreover, if $p_a > 0$ for object $a \in G$, then*

$$\sum_{i\in N}\sum_{B\in\mathbf{B}} n_a(B)x(i,B) = n_a.$$

Proof. The complete proof is in appendix E. The proof is similar to the proof of the existence of Walrasian prices in the unit-demand environment.

□

I now construct the $\mathbf{P-CEEI}$ mechanism. If x^* is a feasible assignment produced by proposition 5.1, then let $\epsilon^* \geq 0$ be the smallest nonnegative real number such that $(1-\epsilon^*)x^*$ is implementable. Set $P-CEEI$ equal to $(1-\epsilon^*)x^*$. Formally, in $P-CEEI$, agents submit their ordinal preferences for acceptable bundles. Then, the designer calculates x^*, using a fake budget $b > 0$ and also finds ϵ^*. Finally, $(1-\epsilon^*)x^*$ is returned as the expected assignments.

5.3 Properties of P-CEEI

A **cardinal mechanism** is a function from the set of cardinal preferences to the set of implementable allocations, $\Delta(\Phi)$. Given an allocation x, let $U_i(x) = \sum_{B\in\mathbf{B}} x(i,B)u_i(B)$; this is agent i's expected utility from mechanism x. Let $U_i(f(u))$ be agent i's expected utility from the mechanism f when the profile of cardinal preferences is u.

Definition 5.3. *A cardinal mechanism f is **asymptotically Pareto-efficient** if, for all $\epsilon > 0$, there exists $q > Q$ such that the following holds:*

For all $q > Q$, there is no implementable allocation y_q such that $\frac{U_i(f(u))}{U_i(y_q)} \leq 1-\epsilon$ for all $i \in N$.

A mechanism is **ex-ante envy-free** if no agent would prefer another agent's assignment. It is **asymptotically ex-post envy-free** if the probability that an agent envies another agent's expected assignment shrinks to zero.

Definition 5.4. *The cardinal mechanism f is ex-ante envy-free if, for all $i, j \in B$ and the profile of cardinal preferences u, the following holds:*

$$U_i(f(u)) \geq \sum_{B\in\mathbf{B}} f(u)(j,B)u_i(B).$$

Tthe cardinal mechanism f is asymptotically ex-post envy-free if, for all $\epsilon > 0$, there exists a $Q > 0$ such that for all $q > Q$, all pairs of agents $i, j \in N_q$, all cardinal-utility profiles, and all bundles $B, B' \in \mathbf{B}^*$, the following holds:

$$B \succ_i B' \;\; \Rightarrow \;\; f_q(u_q)(i,B'),\; f_q(u_q)(j,B') \leq \epsilon.$$

Theorem 3. *P-CEEI is ex-ante envy-free, asymptotically ex-post envy-free, and asymptotically Pareto-efficient.*

Proof. See appendix F.

□

6 Discussion and Conclusion

This paper introduces a probabilistic mechanism (BPS), which generalizes the PS mechanism (Bogomolnaia and Moulin (2001)) to the case of multi-unit demand agents. BPS produces an allocation that is ordinally efficient, asymptotically envy-free, and asymptotically strategy-proof. The difficulty in generalizing the PS mechanism to the case of agents with multi-unit demand is that, unlike the case of agents with single-object demand, feasibility and implementability do not coincide. Therefore, any probabilistic mechanism for the case of agents with multi-unit demand must bridge this gap between feasibility and implementability. Interestingly, as shown in this paper, this issue becomes less severe as the size of the economy increases. The most plausible alternative mechanism that handles general preferences over bundles is the A-CEEI (Budish (2011). Note that A-CEEI, unlike BPS, requires a complex computation of a fixed point.

I have shown that BPS produces an allocation that is asymptotically envy-free. The envy-free property implies that for all cardinal preferences compatible with their ordinal preferences and expected utility, agents prefer their own assignment to that of other agents. However, there could be envy ex-post. In particular, in environments in which agents prefer larger bundles but there is a shortage of objects, agents point to very large bundles at the beginning of the algorithm and would have to point to smaller bundles in the last steps. Therefore, ex-post unfairness is inevitable. To find an ordinal mechanism that handles this issue, one could limit the size of bundles to which agents point. Another option would be is to ask them to point to multiple smaller bundles and to control the relative number of bundles that agents receive ex-post.

Because BPS may induce large ex-post envy, this paper (in section 5) also introduces a cardinal mechanism, P-CEEI, which does better than BPS in the sense of low probability of ex-post envy. P-CEEI is essentially a generalization of CEEI. One can think of P-CEEI as a randomized version of A-CEEI except for the fact with P-CEEI, one does not need to increase the number of objects. Also, P-CEEI is cardinal whereas A-CEEI is ordinal.

In practice, the form of complementarities we might observe could be limited. For example, in the problem of assigning couple interns to hospital residencies complementarities might arise from geographical constraints. In course scheduling, courses can be partitioned into subsets such that courses in the same subset are substitutes for each other and complementarities exist only between courses in different subsets. A possible future research direction is to examine restrictions on complementarities in agents' preferences which would allow one to reduce the gap between feasibility and implementability, to and design mechanisms with attractive properties.

7 Appendices

7.1 Appendix A: proof of proposition 2.3

I show the following lemmas:

Lemma 7.1. *An assignment x is implementable if and only if for a sequence $(\delta_\mu)_{\mu\in\Phi}$ of real numbers, the following inequalities hold:*

- *For all $\omega \in N \times \mathbf{B}$, $x(\omega) \leq \sum_{\mu:\mu(\omega)=1} \delta_\mu$.*
- $\sum_{\mu\in\Phi} \delta_\mu \leq 1$.
- $\delta_\mu \geq 0$ *for all* $\mu \in \Phi$.

Proof. If x is implementable, then a non-negative sequence that satisfies the first two inequalities with equality exists.

To show the reverse, let $(\delta_\mu)_{\mu\in\Phi}$ be the sequence for expected assignment x. If for all $\omega \in N \times \mathbf{B}$, $x(\omega) = \sum_{\mu:\mu(\omega)=1} \delta_\mu$, then $x = \sum_{\mu\in\Phi} \delta_\mu \mu$. This shows that x is implementable. Otherwise, let $\omega \in N \times \mathbf{B}$ be such that the inequality $x(\omega) < \sum_{\mu:\mu(\omega)=1} \delta_\mu$ holds. Let $0 < \alpha_\omega \leq 1$ be such that the equality $x(\omega) = (1-\alpha_\omega)\sum_{\mu:\mu(\omega)=1} \delta_\mu$ holds. For each $\mu \in \Phi$ such that: $\mu(\omega) = 1$, let $\mu_\omega \in \Phi$ be such that the following holds:$\mu_\omega(\omega') = \mu_\omega(\omega')$ iff $\omega' \in N \times \mathbf{B} - \{\omega\}$. I replace $(\delta_\mu)_{\mu\in\Phi}$ with $(\delta'_\mu)_{\mu\in\Phi}$. Where $(\delta'_\mu)_{\mu\in\Phi}$ is constructed as follows: For for all

$$\forall \mu \in \phi \text{ let } \delta'_\mu = \begin{cases} (1-\alpha_\omega)\delta_\mu & \text{if } \mu(\omega) = 1 \\ \delta_\mu + \alpha_\omega \delta'_{\mu'_\omega} & \text{if } \mu(\omega) = 0 \text{ and } \exists \mu' \in \Phi : \mu'_\omega = \mu \\ \delta_\mu & \text{otherwise} \end{cases}.$$

Note that for all $\omega' \in N \times \mathbf{B} - \{\omega\}$, (i) $x(\omega) = \sum_{\mu:\mu(\omega)=1} \delta'_\mu$ and (ii) $\sum_{\mu:\mu(\omega)=1} \delta'_\mu = \sum_{\mu:\mu(\omega)=1} \delta_\mu$. By repeating this modification for all $\omega \in N \times \mathbf{B}$ that satisfy $x(\omega) < \sum_{\mu:\mu(\omega)=1} \delta_\mu$, I construct a sequence $(\bar{\delta}_\mu)_{\mu\in\Phi}$ such that for all $\omega \in N \times \mathbf{B}$, $x(\omega) = \sum_{\mu:\mu(\omega)=1} \bar{\delta}_\mu$.

□

The next lemma is standard.

Lemma 7.2. *Let $P = \{(\overrightarrow{x}, \overrightarrow{y}) \in \mathbb{R}^n \times \mathbb{R}^m| : A\overrightarrow{x} + B\overrightarrow{y} \leq \overrightarrow{b}\}$, where $\overrightarrow{b} \in \mathbb{R}^m$, A is a $m \times n$ matrix, and B is a $m \times k$ matrix. Assume $P \neq \emptyset$. Let $Q = \{\overrightarrow{x} \in \mathbb{R}^n | \exists \overrightarrow{y} \in \mathbb{R}^k$ with $(\overrightarrow{x}, \overrightarrow{y}) \in P\}$. Suppose that $\overrightarrow{u}B =$, $\overrightarrow{u} \geq 0$ has a non-trivial solution. Define Q as follows:*

$$x \in Q \textit{ if } \forall \overrightarrow{u} \in \mathbb{R}^m, \textit{ such that } \overrightarrow{u}B = \overrightarrow{0} \textit{ and } \overrightarrow{u} \geq \overrightarrow{0} \textit{ then } \overrightarrow{u}A\overrightarrow{x} \leq \overrightarrow{u}\overrightarrow{b}.$$

Proof. Let $Q' = \{\overrightarrow{x} \in \mathbb{R}^n | \overrightarrow{u}A\overrightarrow{x} \leq \overrightarrow{u}\overrightarrow{b}, \forall \overrightarrow{u} \in \mathbb{R}^m$ such that $\overrightarrow{u} \geq \overrightarrow{0}$ and $\overrightarrow{u}B = 0\}$. Note that $Q \subseteq Q'$; I show $Q' \subseteq Q$. Let $\overrightarrow{x}^* \notin Q$, then the equation $B\overrightarrow{y} \leq \overrightarrow{b} - A\overrightarrow{x}^*$ does not have any solution $\overrightarrow{y} \in \mathbb{R}^m$. Farkas' lemma implies there exists a $\overrightarrow{u} \in \mathbb{R}^m$ such that $\overrightarrow{u}B = 0$, $\overrightarrow{u} \geq 0$, and $\overrightarrow{u}(A - B\overrightarrow{x}^*) \leq 0$. This implies $\overrightarrow{x}^* \notin Q'$, therefore $Q' \subseteq Q$.

□

The inequalities in lemma 7.1 can be rewritten in the following form:

- $\overrightarrow{x} \leq U\overrightarrow{\delta}$.
- $\overrightarrow{1}.\overrightarrow{\delta} \leq 1$.

- $\overrightarrow{\delta} \geq \overrightarrow{0}$.

  Where $\overrightarrow{\delta} = (\delta_{\phi_1}, \delta_{\phi_2}, \delta_{\phi_3}, ..., \delta_{\phi_f})$. I can rearrange these in the following form:

$$A\overrightarrow{x} + B\overrightarrow{\delta} \leq C,$$

  where

$$A = \begin{bmatrix} I_{p\times p} \\ 0_{1\times p} \\ 0_{f\times p} \end{bmatrix}, \ B = \begin{bmatrix} -U_{p\times f} \\ \overrightarrow{1_f} \\ -I_{f\times f} \end{bmatrix}, \text{ and } C = \begin{bmatrix} 0_{p\times 1} \\ \overrightarrow{1_1} \\ 0_{f\times 1} \end{bmatrix}.$$

By lemma 2, the set of implementable assignments' corresponding vectors can be rewritten as:

$\{\overrightarrow{x} \in \mathbb{R}^P | \overrightarrow{\lambda}.\overrightarrow{x} \leq f \ \ \forall \overrightarrow{\lambda} \in (\mathbb{R}^+ \cup \{0\})^p, \ f \in \mathbb{R}^+ \cup \{0\}, \ \overrightarrow{g} \in (\mathbb{R}^+ \cup \{0\})^f : \ -\overrightarrow{\lambda} U + f\overrightarrow{1} + \overrightarrow{g} = 0\}$.

Dividing by f, I rewrite in the following form:

$$\{\overrightarrow{x} \in \mathbb{R}^P | \overrightarrow{\lambda}.\overrightarrow{x} \leq 1, \ \forall \overrightarrow{\lambda} \in \mathbb{P}^P \text{ such that } \overrightarrow{\lambda} \geq 0 \text{ and } \overrightarrow{\lambda} U \leq \overrightarrow{1}\}.$$

7.2 Appendix B: proof of theorem 1

Let $\succ$ be a preference profile and x be the expected assignment produced by the BPS mechanism. Assume for a contradiction that for some implementable assignment y, all agents prefer, and some strictly, y over x under the FOSD criterion. That is, for all $(i, B) \in N \times \mathbf{B}$:

$$\sum_{S \succeq_i B} x(i, S) \quad \leq \quad \sum_{S \succeq_i B} y(i, S), \tag{1}$$

with strict inequality for some (i, B).

If agent i is allocated bundle B with a positive probability, let $v(i, B)$ be the last step in which agent $i \in N$ is allocated bundle $B \in \mathbf{B}$. It is the smallest v such that $M_v(0)(i, B) = BPS(i, B)$. If agent i is not allocated bundle B, let $v(i, B) = v(i, B')$, where B' is the least preferred bundle that i prefers over bundle B and is allocated with a positive probability. Formally:

$$BPS(i, B') > 0 \text{ and } \forall B'' \in \mathbf{B} \text{ such that } B' \succ_i B'' \succ_i B \text{ then } BPS(i, B'') = 0.$$

Let $(i^*, B^*) \in N \times \mathbf{B}$ be the agent-bundle pair for which inequality (1) holds strict and $v(i^*, B^*)$ is minimal. In other words,

$$(i^*, B^*) = argmin_{\{(i,B) \in N \times \mathbf{B} | \sum_{S \succeq_i B} x(i,S) < \sum_{S \succeq_i B} y(i,S)\}} v(i, B).$$

Let $v^* = v(i^*, B^*)$. For all $v \in \mathbb{N}$, let $\Omega(v) \subseteq N \times \mathbf{B}$ be the set of all agent-bundle pairs whose allocation ended in step v or before, that is, all $(i, B) \in N \times \mathbf{B}$ such that $v(i, B) \leq v$. All bundles that have been allocated before step v^* have the same marginal probability in x and y; that is, $x(\omega) = y(\omega)$ for all $\omega \in \Omega(v^* - 1)$. A constraint from proposition 2.3 has stopped i^* from consuming bundle B^* in step v^*. This constraint has the following form, for some $\lambda : N \times \mathbf{B} \to [0, 1]$ satisfying the conditions in proposition 2.3, the following holds:

$$\sum_{\omega \in N \times \mathbf{B}} \lambda(\omega) x(\omega) \leq 1 \text{ where } \lambda(i^*, B^*) \neq 0 \text{ and } \lambda(\omega) = 0 \ \forall \omega \notin \Omega(v^*).$$

This constraint must hold with equality, since in BPS agents are allocated probability shares of bundles until one of the constraints from proposition 3.2 binds. By construction of v^*, $x(\omega) \leq y(\omega)$ for all $\omega \in \Omega(v^*)$ and $x(i^*, B^*) < y(i^*, B^*)$. This contradicts the implementability of y, since $\sum_{\omega \in \Omega} \lambda(\omega) y(\omega) > 1$.

7.3 Appendix C: proof of proposition 4.4

Let $\mathbf{B}^*$ =$\{S_1, S_2, S_3, ..., S_m\}$. For all $q \in \mathbb{N}$ and $1 \leq j \leq m$, let $c_j^q = \sum_{i \in N_q} x^q(i, S_j)$; this denotes the sum of marginal probabilities that agents are allocated from bundle S_j.

Let $R = \max_{a \in G} \sum_{j=1}^m n_a(S_j)$ and $M \in \mathbb{N}$ be such that $\frac{R+1}{M} \leq \epsilon$ and $M > R$. I can select Q large enough such that the number of objects in the $q-$economy is more than $2M \times R$ for all $q > Q$. For all $1 \leq j \leq m$ and $q > Q$, if $c_j^q \leq M$ set $d_j^q = \lceil c_j^q \rceil$; otherwise, set $d_j^q = \lfloor c_j^q \rfloor - R$. Expected assignment z^q is constructed as $z^q(i, S_j) = \frac{d_j^q}{c_j^q} x^q(i, S_j)$ for all $i \in N_q$ and $1 \leq j \leq m$, and $z^q(\omega) = 0$ for all other $\omega \in N \times \mathbf{B}^*$.

Consider an economy where each S_j is regarded as a single object with d_j^q copies for all $1 \leq j \leq m$. Since for all $1 \leq j \leq m$, $\sum_{i \in N_q} z^q(i, S_j) = d_j^q$, one can implement z^q such that the number of agents that are allocated bundle S_j does not exceed d_j^q. Let $z^q = \sum_{\alpha \in \Lambda^q} p_\alpha z_\alpha^q$, where z_α^q is a deterministic assignment and p_α is the probability of z_α^q. I argue all z_α^q are also implementable in the $q-$economy. If for all $1 \leq j \leq m$, $c_j^q \leq M$, then $\sum_{j=1}^m d_j^q n_a(S_j) \leq M \times R \leq n_a^q$. Otherwise,

$$\begin{aligned}
&\sum_{j=1}^m d_j^q n_a(S_j) \leq \\
&\sum_{j: c_j^q \leq M} (c_j^q + 1) n_a(S_j) + \sum_{j: c_j^q > M} (c_j^q - R) n_a(S_j) \leq \\
&\sum_{j=1}^m c_j^q n_a(S_j) + \sum_{j=1}^m n_a(S_j) - R \leq n_a^q. \qquad (2)
\end{aligned}$$

The last inequality follows from the definition of R: $\sum_{j=1}^m n_a(S_j) \leq R$ for all objects $a \in G$. Inequality (2) guarantees the implementability of z_α^q in the original $q-$economy, hence, z^q is also implementable. For all $i \in N_q$ and $1 \leq j \leq m$, if $c_j^q > M$, then $|x^q(i, S_j) - z^q(i, S_j)| = |\frac{d_j^q - c_j^q}{c_j^q}| x^q(i, S_j) \leq \frac{R+1}{M} \leq \epsilon$. If $c_j^q \leq M$ since $z^q(i, S_j) \geq x^q(i, S_j)$, then $z^q(i, S_j)$ can be reduced to $x^q(i, S_j)$ without harming the implementability of z^q. Therefore, the implementable expected assignment y^q can be constructed such that for all $\omega \in N_q \times B^*$: $|x^q(\omega) - y^q(\omega)| \leq \epsilon$.

7.4 Appendix D: proof of theorem 2

To proceed with the proof, I first prove two lemmas, then I define symmetric implementation, and after that I synchronize the mechanisms with a clock.

Lemma 7.3. *Let $A_{m \times n}$ be a nonnegative integer matrix and $\overrightarrow{b} \in \mathbb{Z}^m$ be a column vector of integers. If the equation $A\overrightarrow{x} \geq \overrightarrow{b}$ has a real solution $\overrightarrow{x} \in \mathbb{R}^n$, then it has a rational solution, formally,*

$$\exists \overrightarrow{x} \in \mathbb{Q}^n \text{ such that } A\overrightarrow{x} \geq \overrightarrow{b}.$$

Proof. Let $\overrightarrow{x^*} \in \mathbb{R}^n$ be the real solution. Since $\mathbb{Q}^n$ is a vector space if the equation does not have any rational solution, Farkas' lemma implies $\exists \overrightarrow{y} \in \mathbb{Q}^m$ such that $\overrightarrow{y} \geq \overrightarrow{0}$, $\overrightarrow{y} A = 0$, and $\overrightarrow{y}\overrightarrow{b} > 0$. Therefore, $\overrightarrow{y}A\overrightarrow{x^*} \geq \overrightarrow{y}\overrightarrow{b}$, which implies $\overrightarrow{y}\overrightarrow{b} \leq 0$. This contradiction proves the lemma.

□

Lemma 7.4. *Let $\overrightarrow{v}_1, \overrightarrow{v}_2, ..., \overrightarrow{v}_k \in (\mathbb{N} \cup \{0\})^n$ and for all $q \in \mathbb{N}$ define the set V_q as follows:*

$$V_q = \{\overrightarrow{x} \in (\mathbb{N} \cup \{0\})^n | \forall\ 1 \leq i \leq k\ ,\ \overrightarrow{v}_i.\overrightarrow{x} \leq c_i^q\},$$

where c_i^q is a non-negative integer number. There exists a uniform bound d such that elements of convex hull of V_q can be represented as a convex combination of elements of V_q with pairwise Euclidean distance of no more than d.

Proof. Proof has three steps. Let W_q be the convex hull of V_q and $||.||$ be the Euclidean norm.

- Step 1) Consider two sequences of vectors $\overrightarrow{x}_q, \overrightarrow{y}_q \in V_q, \forall q \in \mathbb{N}$. I show, there exists two sequences $\overrightarrow{a}_q, \overrightarrow{b}_q \in V_q$ such that: (i) $\overrightarrow{x}_q + \overrightarrow{y}_q = \overrightarrow{a}_q + \overrightarrow{b}_q$ and (ii) $||\overrightarrow{a}_q - \overrightarrow{b}_q||$ is bounded.

  Proof. *Let $(\overrightarrow{a}_q)_{q\in\mathbb{N}}$ & $(\overrightarrow{b}_q)_{q\in\mathbb{N}}$ be two sequences such that: (i) $\overrightarrow{a}_q$ & $\overrightarrow{b}_q \in V_q$ (ii) $\overrightarrow{a}_q + \overrightarrow{b}_q = \overrightarrow{x}_q + \overrightarrow{y}_q$, and (iii) $||\overrightarrow{a}_q - \overrightarrow{b}_q||$ is minimal. If $||\overrightarrow{a}_q - \overrightarrow{b}_q||$ is bounded, we are done; otherwise, let $\overrightarrow{a}_q = (a_q^1, a_q^2, a_q^3, ..., a_q^n)$, $\overrightarrow{b}_q = (b_q^1, b_q^2, b_q^3, ..., b_q^n)$, and $\overrightarrow{a}_q - \overrightarrow{b}_q = (z_q^1, z_q^2, z_q^3, ..., z_q^n)$. For all $1 \leq i \leq k$, let $z_q^{n+i} = \overrightarrow{v}_i.(\overrightarrow{a}_q - \overrightarrow{b}_q)$. One can pick a subsequence and rename the indexes such that $\lim_{q\to\infty} z_q^i \in \mathbb{R}\bigcup\{-\infty\}\bigcup\{+\infty\}$ and $\lim_{q\to\infty} \frac{z_i^q}{z_j^q} \in \mathbb{R}\bigcup\{-\infty\}\bigcup\{+\infty\}$, $\forall 1 \leq i, j \leq n+k$. Since the distance is not bounded, some z_i converges to infinity. Let $1 \leq t \leq n+k$ be the index such that $|z_q^t|$ converges to infinity fastest, in other words, let t be such that $\lim_{q\to\infty} \frac{|z_q^i|}{|z_q^t|} \in \mathbb{R}$ for all $1 \leq i \leq k$. Let $z_i = \lim_{q\to\infty} \frac{z_q^i}{|z_q^t|}$ for all $1 \leq i \leq k+n$, and $\overrightarrow{z} = (z_1, z_2, z_3, ..., z_n)$. By construction of $\overrightarrow{z}$, $\overrightarrow{v}_i.\overrightarrow{z} = z_{n+i}$. If $\overrightarrow{z}$ is a zero vector, then $z_{n+i} = 0$ for all $1 \leq i \leq k$, which contradicts with $z_t \neq 0$; therefore, $\overrightarrow{z}$ is a non-zero vector. Since for all $1 \leq i \leq n$ $\overrightarrow{v}_i$ is an integer vector, applying the previous lemma, one can replace $(z_i)_{1\leq i\leq n+i}$ with rational numbers such that zero elements stay zero and non-zero elements keep their signs. Multiplying by a common denominator ensures that the replacement is an integer vector. Let $\overrightarrow{z}' = (z_1', z_2', z_3', ..., z_{n+k}')$ be the replacement vector and $\hat{\overrightarrow{z}} = (z_1', z_2', z_3', ..., z_n')$. Since $\overrightarrow{z}$ is a non-zero vector, $\hat{\overrightarrow{z}}$ is also a non-zero vector. Let $\overrightarrow{x}'_q = \overrightarrow{a}_q - \hat{\overrightarrow{z}}$ and $\overrightarrow{y}'_q = \overrightarrow{b}_q + \hat{\overrightarrow{z}}$. I show for large enough q, $\overrightarrow{x}'_q$ & $\overrightarrow{y}'_q \in V_q$. For all $1 \leq i \leq n$, if $\hat{z}_i \leq 0$, then $a_q^i \geq 0 \geq \hat{z}_i$. If $\hat{z}_i > 0$, then $\lim_{q\to\infty} a_q^i = \infty$; hence for large enough q, $a_q^i > \hat{z}_i$, therefore, $\overrightarrow{x}'_q \in (\mathbb{N} \cup \{0\})^n$ for large enough q. For all $1 \leq i \leq k$, I have $\overrightarrow{v}_i.\overrightarrow{x}'_q = \overrightarrow{v}_i.\overrightarrow{a}_q - \overrightarrow{v}_i\hat{z} = \overrightarrow{v}_i.\overrightarrow{a}_q - z_{n+i}'$. If $z_{n+i}' \geq 0$ since $\overrightarrow{v}_i.\overrightarrow{a}_q \leq c_i^q$, then $\overrightarrow{v}_i.\overrightarrow{x}'_q \leq c_i^q$; if $z_{n+i}' < 0$, then $z_{n+i} < 0$. Therefore, $\lim_{q\to\infty} \overrightarrow{v}_i.(\overrightarrow{a}_q - \overrightarrow{b}_q) = -\infty$, which implies $\lim_{q\to\infty} \overrightarrow{v}_i.\overrightarrow{a}_q - c_i^q = -\infty$. Hence, for large enough q, $\overrightarrow{v}_i.\overrightarrow{x}'_q \leq c_i^q$. This shows for large enough q, $\overrightarrow{x}'_q \in V_q$. A similar argument shows $\overrightarrow{y}'_q \in V_q$ for large enough q. If $\hat{z}_i > 0$, then $\lim_{q\to\infty} z_q^i = +\infty$, and if $\hat{z}_i < 0$, then $\lim_{q\to\infty} z_q^i = -\infty$. Therefore, if $\overrightarrow{x}'_q - \overrightarrow{y}'_q = (z_q^1 - 2\hat{z}_1, z_q^2 - 2\hat{z}_2, z_q^3 - 2\hat{z}_3, ..., z_q^n - 2\hat{z}_n)$ then for all $1 \leq i \leq n$, $|z_q^i - 2\hat{z}_i| \leq |z_q^i|$, with strict inequality for some i. This implies for large enough q, $||\overrightarrow{x}'_q - \overrightarrow{y}'_q|| < ||\overrightarrow{a}_q - \overrightarrow{b}_q||$, which contrasts the minimality of $||\overrightarrow{a}_q - \overrightarrow{b}_q||$, since for all q the equality $\overrightarrow{x}'_q + \overrightarrow{y}'_q = \overrightarrow{a}_q + \overrightarrow{b}_q$ holds.*

- Step 2) Let $\overrightarrow{x}_q, \overrightarrow{y}_q \in \mathbb{R}^n$ be two sequences of vectors such that $||\overrightarrow{x}_q - \overrightarrow{y}_q||$ converges to infinity. Let $\overrightarrow{m}_q = \frac{\overrightarrow{x}_q + \overrightarrow{y}_q}{2}$ and $\overrightarrow{m}'_q \in \mathbb{R}^n$ be a sequence with a bounded distance from $\overrightarrow{m}_q$. For large enough q and all $\overrightarrow{R} \in \mathbb{R}^n$: $||\overrightarrow{R} - \overrightarrow{m}'_q|| < \max\{||\overrightarrow{R} - \overrightarrow{x}_q||, ||\overrightarrow{R} - \overrightarrow{y}_q||\}$.

  Proof. *Note that* $||\overrightarrow{R} - \overrightarrow{m}_q||^2 = \frac{||\overrightarrow{R} - \overrightarrow{x}_q||^2}{2} + \frac{||\overrightarrow{R} - \overrightarrow{y}_q||^2}{2} - \frac{||\overrightarrow{x}_q - \overrightarrow{y}_q||^2}{4}$. *Since* $||\overrightarrow{m}_q - \overrightarrow{m}'_q|| = O(1)$ *and* $\frac{||\overrightarrow{x}_q - \overrightarrow{y}_q||^2}{4}$ *converges to infinity, for large enough* q *I have:*

$$
\begin{aligned}
&||\overrightarrow{R} - \overrightarrow{m}'_q|| < ||\overrightarrow{R} - \overrightarrow{m}_q|| + ||\overrightarrow{m}_q - \overrightarrow{m}'_q|| = \\
&\sqrt{\frac{||\overrightarrow{R} - \overrightarrow{x}_q||^2}{2} + \frac{||\overrightarrow{R} - \overrightarrow{y}_q||^2}{2} - \frac{||\overrightarrow{x}_q - \overrightarrow{y}_q||^2}{4}} + O(1) \\
&< \sqrt{\frac{||\overrightarrow{R} - \overrightarrow{x}_q||^2}{2} + \frac{||\overrightarrow{R} - \overrightarrow{y}_q||^2}{2}} \\
&< \max\{||\overrightarrow{R} - \overrightarrow{x}_q||, ||\overrightarrow{R} - \overrightarrow{y}_q||\}.
\end{aligned}
$$

- Step 3) If the lemma is not true, there exists an increasing sequence of integers, $(q_t)_{t=1}^{\infty}$, and a sequence $\overrightarrow{s}_{q_t} \in W_{q_t}$ which is in the convex hull of the elements of the following set:

$$
A_t = \{\overrightarrow{x}^t_1, \overrightarrow{x}^t_2, \overrightarrow{x}^t_3, ..., \overrightarrow{x}^t_{a_t}\} \subset Vq_t,
$$

  where the diameter of A_t converges to infinity. The diameter of a set is the largest distance of two points in that set. Let $f(A_t)$ be the sum of pairwise distance of points in A_t with maximal distance, formally:

$$
f(A_t) = [\#\{(i,j) | 1 \leq i, j \leq n,\ ||\overrightarrow{x}_i - \overrightarrow{x}_j|| = \max_{i,j} ||\overrightarrow{x}_i - \overrightarrow{x}_j||\}]. \max_{i,j} ||\overrightarrow{x}_i - \overrightarrow{x}_j||.
$$

  Without loss of generality, one can pick A_t such that the diameter of A_t is minimal and among those with minimal diameter, $f(A_t)$ is minimal. Without loss of generality, I assume the diameter of A_t is equal to $||\overrightarrow{x}^t_1 - \overrightarrow{x}^t_2||$. From step 1, there are sequences $\overrightarrow{y}^t_1$ and $\overrightarrow{y}^t_2$ with bounded distance such that for all q: $\overrightarrow{x}^t_1 + \overrightarrow{x}^t_2 = \overrightarrow{y}^t_1 + \overrightarrow{y}^t_2$. Let $\overrightarrow{s}_t = \sum_{i=1}^{a_t} \alpha^t_i \overrightarrow{x}^t_i$, if $\alpha^t_1 \leq \alpha^t_2$, then set $A'_t = A_t \cup \{\overrightarrow{y}^t_1\} \cup \{\overrightarrow{y}^t_2\} \setminus \{\overrightarrow{x}^t_1\}$ and if $\alpha^t_2 > \alpha^t_1$, then set $A'_t = A_t \cup \{\overrightarrow{y}^t_1\} \cup \{\overrightarrow{y}^t_2\} \setminus \{\overrightarrow{x}^t_2\}$. Note that $\overrightarrow{s}_t$ is in the convex hull of A'_t. Since $\overrightarrow{y}^t_1$ and $\overrightarrow{y}^t_2$ have bounded distance, they have a bounded distance from their midpoints as well. Step 2 implies that for large enough t, $i = 1, 2$, and $\overrightarrow{R} \in \mathbb{R}^n$, $||\overrightarrow{R} - \overrightarrow{y}^t_i|| < \max\{||\overrightarrow{R} - \overrightarrow{x}^t_1||, ||\overrightarrow{R} - \overrightarrow{x}^t_2||\}$. This shows the diameter of A'_t is no more than the diameter of A_t for large enough t. Since the diameter of A_t is minimal, the diameter of A'_t is equal to the diameter of A_t, for large enough t. Assume t is large enough such that the diameters of A_t and A'_t are equal. I show $f(A'_t) < f(A_t)$. Since $||\overrightarrow{R} - \overrightarrow{y}^t_i|| < \max\{||\overrightarrow{R} - \overrightarrow{x}^t_1||, ||\overrightarrow{R} - \overrightarrow{x}^t_2||\}$, distance of y^t_i from other points in A'_t is less than the diameter of A'_t for $i = 1, 2$. Since $\{\overrightarrow{x}^t_1, \overrightarrow{x}^t_2\} \not\subseteq A'_t$, the number of pairs with distance equal to the diameter is smaller in A'_t compared to A_t. This shows $f(A'_t) < f(A_t)$ which contradicts the minimality of $f(A_t)$.

  □

I define **symmetric implementation**. For any deterministic assignment in the $q-$economy, a permutation of this assignment is a deterministic assignment in which agents with the same type switch their assigned bundles. An implementation is called symmetric if for any deterministic assignments that appear in the implementation, all of it's permutations also appear with the same probability. It is easy to see that all symmetric implementable assignments can be implemented symmetrically. Let $\mathbf{B}^* = \{S_1, S_2, S_3, ..., S_m\}$. Consider a deterministic implementable assignment x in the $q-$economy. For all $1 \le \theta \le k$ and $1 \le j \le m$, let $n_{\theta,j}$ be the number of agents with type θ that are allocated bundle S_j. I call $M = [n_{\theta,j}]_{k\times m}$ the **corresponding matrix** of x. Implemetability of x is equivalent to:

- $\sum_{j=1}^{m} n_{\theta,j} \le |\Pi_\theta|$.
- $\sum_{\theta=1}^{k}\sum_{j=1}^{m} n_{\theta,j} n_a(S_j) \le n_a$ for all objects $a \in G$.

A uniform distribution over all deterministic allocations with corresponding matrix M produces an expected assignment in which agents with type θ consume bundle S_j with probability $\frac{n_{\theta,j}}{|\Pi_\theta^q|}$. An additional randomization over corresponding matrices $(M^o)_{o\in I}$ with probability distribution $(f_o)_{o\in I}$ produces an expected assignment in which agents with type θ consume bundle j with probability $\frac{\sum_{o\in I} f_o n^o_{\theta,j}}{|\Pi^q_\theta|}$. The distance between two deterministic allocations with corresponding matrices $M_1 = [m^1_{\theta,j}]$ and $M_2 = [m^2_{\theta,j}]$ is the largest array in $|M_1 - M_2|$; in other words, $\max_{\theta,j} |m^1_{\theta,j} - m^2_{\theta,j}|$. Lemma 5.4 shows for some d, all symmetric implementable assignments in the $q-$economy can be implemented via deterministic allocations with distance at most d. Corresponding matrix of a deterministic allocation can be naturally defined in the continuum economy. This form of implementation and the corresponding matrix can be natural generalized to the continuum economy. Given a deterministic assignment x^* in the continuum economy, $n^*_{\theta,j}$ is the measure of agents with type θ that are allocated bundle S_j.

To proceed with the proof, I **synchronize** the BPS and the NPS mechanisms to a clock. The date at which a bundle B is consumed by agent i in the BPS mechanism is a time interval that I represent with $[t^-(i,B), t^+(i,B)]$. To calculate the boundaries of the interval consider two cases:

- i) Agent $i \in N$ consumes bundle $B \in \mathbf{B}$ with a positive probability, i.e., $BPS(i,B) > 0$. Let $\epsilon_{M_{-1}(0)} = 0$, then $t^-(i,B) = \sum_{\rho=0}^{t-1} \epsilon_{M_{\rho-1}(0)}$ and $t^+(i,B) = \sum_{\rho=0}^{s} \epsilon_{M_{\rho-1}(0)}$, where t is the first step in the $M_1(0), M_2(0), M_3(0), ...$ sequence in which agent i consumes bundle B and s is the last step, i.e. $M_t(0)(i,B) > 0$, $M_{t-1}(0)(i,B) = 0$, and $\forall\ s \ge t$ $M_s(0)(i,B) = M_{s+1}(0)(i,B)$.
- ii) Agent $i \in N$ does not consume bundle $B \in \mathbf{B}$ with a positive probability, i.e., $BPS(i,B) = 0$. Let S be i's least preferred bundle among all bundles that he prefers over bundle B and the mechanism allocates to i with a positive probability, i.e., the least preferred bundle that satisfies $BPS(i,S) > 0$ and $S \succ_i B$. If t is the last step in which agent i consumes bundle S, $t^-(i,B) = t^+(i,B) = \sum_{\rho=0}^{t} \epsilon_{M_{\rho-1}(0)}$.

For $1 \le \theta \le k$ and $q \in \mathbb{N}$, let $[t^-_q(\theta,B), t^+_q(\theta,B)]$ be the time interval in which agents with type θ in the $q-$economy are allocated bundle $B \in \mathbf{B}^*$ in the BPS mechanism.

I define the time intervals for the $NPS^*(= BPS^*)$ mechanism similarly. The date in which agents with type θ consume bundle $B \in \mathbf{B}^*$ in the NPS^* mechanism is

$$[s^-(\theta,B), s^+(\theta,B)],$$

where $s^-(\theta,B) = \min_v\{t^*(v) | m^*_i(G^*(v)) = B\}$ and $s^+(\theta,B) = \max_v\{t^*(v)|\ m^*_\theta(G^*(v-1)) = B\}$. Given a symmetric mechanism f, let $f^q(\theta,B)$ and $f^*(\theta,B)$ be the marginal probabilities that agents with type θ are given bundle B. It is easy to see that $NPS^*(\theta,B) =$

$s^+(\theta, B) - s^-(\theta, B)$ and $BPS^q(\theta, B) = t_q^+(\theta, B) - t_q^-(\theta, B)$.

I show for all $1 \le \theta \le k$ and $B \in \mathbf{B}^*$, $\lim_{q\to\infty} t_q^-(\theta, B) = s^-(\theta, B)$ and $\lim_{q\to\infty} t_q^+(\theta, B) = s^+(\theta, B)$. This proves the theorem. I show the equality holds by induction on $s^+(\theta, B)$. The smallest $s^+(\theta, B)$ is the time of the first step in the NPS^* mechanism. Since NPS^q converges to NPS^* and in both NPS^q and BPS^q agents are initially allocated probability shares of their best bundles, rounding proposition implies the induction base. The induction hypothesis is that the result is true for all (θ, B) that end prior to step v in NPS^*, that is $s^+(\theta, B) \le t(v)$. Let $\Omega^*(v+1)$ be the set of $(\theta, B) \in \{1, 2, 3, ..., \theta\} \times \mathbf{B}^*$ for which the NPS^* mechanism stops allocating probability shares of bundle B to agents with type θ in step $v+1$. Since implementability implies feasibility, it cannot be that $\limsup_{q\to\infty} t_q^+(\omega) > t^*(v+1)$ for all $\omega \in \Omega^*(v+1)$. Let τ be the smallest limit of ending dates consumption of these bundles, i.e., $\tau = \min_{\omega\in\Omega^*(v+1)} \limsup_{q\to\infty} t_q^+(\omega)$. Showing that $\tau = t^*(v+1)$ completes the proof. If $\tau < t^*(v+1)$ since $\tau \ge t^*(v)$, then $t^*(v) < t^*(v+1)$. Let $BPS^q(t)$ and $NPS^*(t)$ be the allocation at time t in the BPS and NPS algorithm. Note that both $BPS^q(t)$ and $NPS^*(t)$ are symmetric expected assignments. Let $BPS^q(t)(\theta, j)$ and $NPS^*(\theta, j)$ be the probabilities that an agent with type θ consumes bundle S_j up to time t in the corresponding economy and mechanism. I prove the following lemma:

Lemma 7.5. *Assume the induction hypothesis. If $\tau < t^*(v+1)$, then for some $\delta > 0$, an increasing sub-sequence of natural numbers $(q_n)_{n=1}^{\infty}$, and all $\epsilon < \delta$ the following expected assignment is not implementable:*

$$BPS^{q_n}(t^*(v)) + \textstyle\sum_{\bar{\omega}\in\Omega^*(v+1)} (t^*(v+1) - t^*(v) - \epsilon) I_{\{\omega=\bar{\omega}\}}.$$

Proof. If the lemma is not true, then for all ϵ that satisfy $t^*(v+1) - t^*(v) > \epsilon > 0$, there exists a Q such that $\forall q > Q$, $BPS^q(t^*(v)) + \sum_{\omega\in\Omega^*(v+1)} (t^*(v+1) - t^*(v) - \epsilon) I_{\{\omega=\bar{\omega}\}}$ is implementable. Since $\tau < t^*(v+1)$, assume $\epsilon < t^*(v+1) - \tau$. This implies $BPS^q(t^*(v)) + \sum_{\bar{\omega}\in\Omega^*(v+1)} (\tau - t^*(v) + \epsilon) I_{\{\omega=\bar{\omega}\}}$ is implementable for some $\epsilon > 0$ and large enough q. This contradicts the definition of τ.

□

Consider the $(q_n)_{n=1}^{\infty}$ sequence from lemma 7.5. Since

$$\lim_{n\to\infty} BPS^{q_n}(t^*(v)) \;\; = \;\; NPS^*(t(v)),$$

for $\epsilon_1 > 0$ there exists $N_1 > 0$ such that for all $1 \le \theta \le k$, $1 \le j \le m$, and $n > N_1$, $|BPS^{q_n}(t^*(v))(\theta, j) - NPS^*(t^*(v))(\theta, j)| < \epsilon_1$. Let $(N_\alpha^{q_n} = [n_{\theta,j}^\alpha]_{k\times m})_{\alpha\in\Delta_n}$ be the set of corresponding matrices in $BPS^{q_n}(t^*(v))$'s implementation. Since it is symmetric, assume matrices are within distance of d. Let $\pi_n(\alpha)$ be the probability of matrix $N_\alpha^{q_n}$. For all $1 \le \theta \le k$, $1 \le j \le m$, and $\alpha, \alpha' \in \Delta_{qn}$, $|n_{\theta,j}^\alpha - n_{\theta,j}^{\alpha'}| < d$. Since $BPS^{q_n}(t^*(v))(\theta, j)$ is a convex combination of $(\frac{n_{\theta,j}^\alpha}{\Pi_\theta^{q_n}})_{\alpha\in\Delta_q}$, it must be that $|BPS^{q_n}(t^*(v))(\theta, j) - \frac{n_{\theta,j}^\alpha}{\Pi_\theta^{q_n}}| < \frac{d}{\Pi_\theta^{q_n}}$. For $\epsilon_2 > 0$, let $N_2 > N_1$ be such that for all $n > N_2$, $\frac{d}{\Pi_\theta^{q_n}} < \epsilon_2$. As the proof of rounding proposition shows, $NPS^*(t(v))$ can be implemented with the corresponding matrix $N^* = [n_{\theta,j}^*]_{k\times m}$ that satisfies $\frac{n_{\theta,j}^*}{\Pi_\theta^*} = NPS^*(t^*(v))(\theta, j)$ for all $1 \le \theta \le k$ and $1 \le j \le m$. If $n > N_2$, then for all $1 \le \theta \le k$, $1 \le j \le m$, and $\alpha \in \Delta_n$, $|\frac{n_{\theta,j}^*}{\Pi_\theta^*} - \frac{n_{\theta,j}^\alpha}{\Pi_\theta^{q_n}}| < \epsilon_1 + \epsilon_2$. Consider the expected assignment $NPS^*(t^*(v+1)) - NPS^*(t^*(v))$, assume $M^* = [m_{\theta,j}^*]_{k\times m}$ is its corresponding matrix designed as in the proof of rounding proposition. Let $M = [m_{\theta,j}^n]_{k\times m}$ be a corresponding matrix in the q_n-economy such that $\frac{m_{\theta,j}^n}{\Pi_\theta^{q_n}} = \frac{m_{\theta,j}^*}{\Pi_\theta^*}$ for all θ and j. Since $NPS^*(t^*(v)) + NPS^*(t^*(v+1)) - NPS^*(t^*(v))$ is implementable in the continuum economy, it is also feasible. Therefore,

- $\sum_{j=1}^{m} n^*_{\theta,j} + m^*_{\theta,j} \leq |\Pi^*_\theta| \Rightarrow \sum_{j=1}^{m} \frac{n^*_{\theta,j}+m^*_{\theta,j}}{|\Pi^*_\theta|} \leq 1$,
- $\sum_{\theta=1}^{k} \sum_{j=1}^{m} \frac{(n^*_{\theta,i}+m^*_{\theta,j})}{\Pi^*_i} n_a(S_j) \leq \frac{q^*_a}{\Pi^*_\theta}$ for all objects $a \in G$.

Choose $\epsilon > 0$ such that $\epsilon < \min_{\theta,j}\{\frac{n^*_{\theta,j}+m_{\theta,j}*}{|\Pi^*_\theta|} | \ n^*_{\theta,j} + m^*_{\theta,j} \neq 0\}$. I construct the following corresponding matrices in the q_n-economy.

For all $1 \leq \theta \leq k$, $1 \leq j \leq m$, $n \in N_2$, and $\alpha \in \Delta_n$, $P^\alpha = [p^\alpha_{\theta,j}]$ is defined as follows:

$$\begin{aligned} &\text{if } m^*_{\theta,j} + n^*_{\theta,j} = 0 \quad \text{set } p^\alpha_{\theta,j} = 0, \\ &\text{if } m^*_{\theta,j} + n^*_{\theta,j} \neq 0 \quad \text{set } p^\alpha_{\theta,j} = n^\alpha_{\theta,j} + m^n_{\theta,j} - \epsilon \Pi^{q_n}_\theta. \end{aligned}$$

Note that for all $\alpha \in \Delta_n$, P^α satisfies the following inequalities:

- for all $1 \leq \theta \leq k$, $\sum_{j=1}^{m} p^\alpha_{\theta,j} \leq 1$,
- for all objects $a \in G$, $\sum_{\theta=1}^{k} \sum_{j=1}^{m} p^\alpha_{\theta,j} n_a(S_j) \leq \Pi^{q_n}_\theta$.

These two inequalities imply a deterministic assignment with corresponding matrix P^α is implementable. Let x^α be an expected assignment that is generated by a uniform randomization over all deterministic assignments with corresponding matrix P^α. Also, let $x = \sum_{\alpha \in \Delta_n} \pi_n(\alpha) x^\alpha$. Note that x is an implementable assignment that satisfies

$$x(\theta, j) \geq BPS^{q_n}(t^*(v))(\theta, j) + \sum_{\bar{\omega} \in \Omega^*(v+1) \cap \{\Pi^{q_n}_\theta \times \{S_j\}\}} (t^*(v+1) - t^*(v) - \epsilon) I_{\{\omega = \bar{\omega}\}}.$$

Since x is implementable for large enough n and small enough ϵ the following expected assignment is also implementable

$$BPS^{q_n}(t^*(v))(\theta, j) + \sum_{\bar{\omega} \in \Omega^*(v+1) \cap \{\Pi^{q_n}_\theta \times \{S_j\}\}} (t^*(v+1) - t^*(v) - \epsilon) I_{\{\omega = \bar{\omega}\}}.$$

This contradicts with lemma 7.5.

7.5 Appendix E: proof of proposition 5.1

The proof is an adaptation of a standard existence of Walrasian price, see Budish et al. (2011).

Given any price vector $p = (p_a)_{a \in G}$, let $X(P)$ be the set of demanded random assignments. Formally,

$$X(p) = \{x : N \times \mathbf{B} \to [0, 1] | \text{ if } \forall i \in N \text{ agent } i \text{ demands bundle } i\}.$$

Given a random assignment x let $e(x) = (e(x)_a)_{a \in G}$ be the excess demand vector of x, i.e.,

$$\forall a \in G \quad e(x)_a = \sum_{i \in N} \sum_{B \in \mathbf{B}} n_a(B) x(i, B) - n_a(G)\}.$$

For each price vector $p \in [0, b]^{|G|}$, define the excess demand correspondence Z that maps a price vector to the set of excess demand vectors. Formally,

$$Z(p) = \{z = (z_a)_{a \in G} | \exists x \in X(p) : \ e(x) = z\}.$$

Expand the price space to $\hat{P} = [-|N|-\kappa|N||G|, |N|+\kappa|N||G|]^{|G|}$. Consider the function $s : \hat{P} \to [0,1]^{|G|}$, where $s(p) = (\max(0, \min(p_a, |N|)))_{a\in G}$. Define the correspondence $y : \hat{P} \to \hat{P}$ as $y(p) = s(p) + Z(s(p))$.[11] This correspondence admits a fixed point, since it is convex and upper hemicontinuous. Let p^* be a fixed point and allocation $x^* \in X(s(p^*))$ be such that $y(p^*) = s(p^*) + e(x^*)$. I show for all $a \in G$, $p^*_a < |N|$. If for some $a \in G$, $p^*_a \geq |N|$, then agents can not afford more than $\frac{1}{|N|}$ probability share of bundles that include object a. Hence, the a component of $e(x^*)$ is negative. Since $s_a(p^*) = |N|$, it implies that $p^*_a < |N|$. This contradiction shows for all $a \in G$ $p^*_a < |N|$. For all $a \in G$, if $p^*_a \geq 0$, then $s_a(p^*) = p^*_a$which implies $e_a(x^*) = 0$. Hence, there is no excess or shortage of demand for object a. If $p^*_a < 0$ then $e_a(x^*) < 0$, which implies there is an excess supply of object a. The price vector $s(p^*)$ and feasible expected assignment x^* satisfy the conditions in the proposition.

7.6 Appendix F: proof of theorem 3

Let $P-CEEI^q = (1-\epsilon^q)x^q$, where x^q is the feasible allocation in the $q-$economy produced according to the proposition 5.1. From the rounding proposition we know for $\epsilon > 0$, there exists a $Q > 0$ such that for all $q > Q$ $\epsilon^q < \epsilon$.

Let x^q_i and x^q_j be agents i and j assignments in the feasible allocation x^q. Their assignments in the $P - CEEI$ mechanism is $(1-\epsilon^q)x^q_i$ and $(1-\epsilon^q)x^q_j$ be agent i and j's assignments. Since both agents were given the same budget, x^q_j is affordable for agent i. Therefore, $U_i(x^q) = U_i(x^q_i) \geq U_i(x^q_j)$. which implies $U_i((1-\epsilon^q)x^q) = U_i((1-\epsilon^q)x^q_i) \geq U_i((1-\epsilon^q)x^q_j)$ this establishes the ex-ante envy-free property. An agent envies another agent only when he is allocated $\emptyset$. Since $P - CEEI^q(i, \emptyset) < \epsilon$ for all $q > Q$ and $i \in N_q$ the mechanism is asymptotically ex-post envy-free.

Let $q > Q$. If $\frac{U_i((1-\epsilon^q)x^q_i)}{U_i(y_q)} \leq 1-\epsilon$ for all $i \in N_q$, since $\epsilon^q < \epsilon$, then $U_i(y_q) > U_i(x^q_i)$ for all $i \in N_q$. Therefore y_q is not affordable for any agent, i.e.,

$$\forall i \in N_q \sum_{B\in\mathbf{B}^*} y^q(i,B)\sum_{a\in G} n_a(B)p_a \quad > \quad 1 \geq \sum_{B\in\mathbf{B}^*} x^q(i,B)\sum_{a\in G} n_a(B)p_a. \tag{3}$$

Note that for all $a \in G$ $p_a = 0$ or $\sum_{i\in N_q}\sum_{B\in\mathbf{B}^*} x^q(i,B)n_a(B) = n^q_a$, therefore

$$\forall i \in N_q \ \sum_{i\in N_q}\sum_{B\in\mathbf{B}^*} x^q(i,B)n_a(B)p_a \quad = p_a n^q_a. \tag{4}$$

Add up 3 for all $i \in N_q$ and apply 4 to imply:

$$\sum_{a\in G}\sum_{i\in N_q}\sum_{B\in\mathbf{B}^*} y^q(i,B)n_a(B)p_a > \sum_{a\in G} p_a n^q_a. \tag{5}$$

The inequality 5 contradicts with the implementability of y^q. Therefore the mechanism is asymptotically efficient.

[11] Given a set $A \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$ let $x + A = \{x + y | y \in A\}$.

References

[1] A. Abdulkadiroglŭ, S. Sönmez, Random Serial Dictatorship and the Core from Random Endowments in House Allocation Problem, Econometrica 66 (1998), 689-701.

[2] A.Bogomolnaia and H. Moulin, A New Solution to the Random Assignment Problem, Journal of Economic Theory 100 (2001), 295-328.

[3] E. Budish, The Combinatorial Assignment Problem: Approximate Competitive Equilibrium from Equal Income, Journal of Political Economy 119 (2011), 1061-1103.

[4] E. Budish, E. Cantillon, The Multi-Unit Assignment Problem: Theory and Evidence from Course Allocation at Harvard, American Economic Review 102 (2012), 2237-71.

[5] E. Budish, Y.K. Che, F. Kojima, and P. Milgrom, Designing Random Allocation Mechanisms: Theory and Applications, American Economic Review 103 (2013), 585-623.

[6] Y.K. Che, F. Kojima, Asymptotic Equivalence of Probabilistic Serial and Random Priority Mechanisms, Econometrica 78 (2010), 1625–1672.

[7] Y. Chen, T. Sönmez, Improving Efficiency of On-campus Housing: An Experimental Study. American Economic Review 92 (2002), 1669-1686.

[8] A.K. Kattaa, J. Sethuraman, A Solution to the Random Assignment Problem on the Full Preference Domain, Journal of Economic Theory 131 (2006), 231–250.

[9] F. Kojima, Random Assignment of Multiple Indivisible Objects, Mathematical Social Sciences 57 (2009), 134–142.

[10] F. Kojima, M. Manea, Incentives in the Probabilistic Serial Mechanism, Journal of Economic Theory 145 (2010), 106–123.

[11] F. Kojima, P. Pathak, A.E. Roth, Matching with Couples: Stability and Incentives in Large Markets, NBER Working Paper No 16028 (2010).

[12] M. Pycia, Ordinal Efficiency, Fairness, and Incentives in Large Markets, working paper (2012), Economics Department, University of California, Los Angeles.

[13] M. Manea, A Constructive Proof of the Ordinal Efficiency Welfare Theorem, Journal of Economic Theory 141 (2008), 276-281.

[14] M. Manea, Asymptotic Ordinal Inefficiency of Random Serial Dictatorship, Theoretical Economics 4 (2009), 165–197.

[15] L. Shapley, H. Scarf, On Cores and Indivisibility, Journal of Mathematical Economics 1 (1974), 23–37.

[16] Varian, Hal R. Equity, Envy, and Efficiency, Journal of Economic Theory 9 (1974), 63-91.